

On phylogenetic trees – a geometer’s view ^{*}

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Abstract: In the present note we investigate projective varieties which are geometric models of binary symmetric phylogenetic 3-valent trees. We prove that these varieties have Gorenstein terminal singularities (with small resolution) and they are Fano varieties of index 4. Moreover any two such varieties associated to trees with the same number of leaves are deformation equivalent, that is, they are in the same connected component of the Hilbert scheme of the projective space. As an application we provide a simple formula for computing their Hilbert-Ehrhard polynomial.

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0 Introduction

Algebraic geometry, a classical, almost ancient, branch of pure mathematics, is constantly stimulated by questions arising in applicable mathematics and other sciences. String theory and mirror conjecture from mathematical physics, coding theory and image recognition from computer sciences — to mention just a few of the big areas of sciences which had enormous impact on the development of algebraic geometry in the past decade. Now the modern biology with its computational aspects and relations to statistics seems to be making its way into this branch of mathematics.

Although the roots of questions which we tackle are beyond the area of our professional interest and we do not claim any thorough understanding of them still, the questions formulated in the language of our trade seem to be extremely interesting for its own, mathematical meaning. In fact, we believe that most of the important things in mathematics are related to real phenomena of Nature. The interpretation of this profound feature of Mathematics is left for the reader and it will definitely depend on the reader's attitude towards fundamental Creation vs. Evolution problem, cf. [Shafarevich] and [Reid '87].

Knowing our limitations as laymen in computational biology and statistics we try to stay within borders of the branch of mathematics which we believe we understand. That is why we take a relatively simple model, redefine it in purely algebraic language and examine it using methods of algebraic geometry. The result exceeds our original expectations, we find the object appearing in this process very interesting for its own, pure geometric aspects, with properties which we have not expected originally.

Our original task was computing Hilbert-Ehrhard polynomials for varieties arising as geometrical models of binary symmetric 3-valent phylogenetic trees. The question is consistent with the attitude of computational algebraic geometry and

algebraic statistics where the point is to compute and understand the ideal of the variety in question in the ambient projective space. Then the Hilbert-Ehrhard polynomial provides a fundamental invariant of such an ideal, the dimensions of homogeneous parts of it. To our surprise the polynomial does not depend on the shape of the tree but merely on its size, the number of leaves or, equivalently the dimension of its geometric model. The strive to understanding this phenomenon lead us to proving one of the main results of the present paper, 3.26, which asserts that models of trees with the same number of leaves are deformation equivalent, that is they are in the same connected component of the Hilbert scheme of the projective space in question (hence they have the same Hilbert polynomial).

The fact that the geometric models of trees modelling some processes — the discreet objects — live in a connected continuous family of geometric objects probably deserves its explanation in terms of algebraic statistic or even biology. For the algebraic geometry part we have a natural question arising about irreducibility of the component of the Hilbert scheme containing these models and (if the irreducibility is confirmed) about varieties which arise as general deformations (that is, over a general point of the component of the Hilbert scheme in question). The question about a general deformation of the model is related to the other main result of the present paper, 3.17, which is that these models are index 4 Fano varieties with Gorenstein terminal singularities. Thus one would expect that their general deformation is a *smooth* Fano variety of index 4, c.f. [Namikawa].

The present paper is organized as follows. We deal with varieties defined over complex numbers. In the first section we define phylogenetic trees and their geometric models. We do it in pure algebraic way and with many simplifications: we deal with unrooted symmetric trees which are then assumed to be binary and eventually 3-valent. From the algebraic geometer point of view studying geometric models in this case can be reduced to understanding special linear subsystems of the Segre linear system on a product of \mathbb{P}^1 's, 1.9. Eventually, the question boils down to studying fixed points of the Segre system with respect to an action of a group of involutions, 1.12. Since the action can be diagonalized this brings us down to toric geometry.

In the second section we define a geometric model of a tree in terms of toric geometry, via a polytope in the space of characters of a complex torus, which we call a polytope model of the tree and to which we subsequently associate a projective variety. The main results of this part are 2.12 and 2.24 which assert that the models defined in the first part are the same as these defined the toric way. In this part we also prove results which are of the fundamental technical importance: this is a fiber product formula for polytopes of trees, 2.20, and its counterpart for varieties,

a quotient formula 2.26. The latter asserts that the geometric model of a tree obtained by gluing two smaller trees is a Mumford’s GIT (Geometric Invariant Theory) quotient of the product of their respective models.

The third section of the present paper contains its main results. After a brief discussion of equations defining a geometric model of a tree, with special consideration to a tree with two inner nodes and four leaves, we examine fans of geometric models and resolution of their singularities. We prove that geometric models of 3-valent binary symmetric trees are index 4 Fano varieties with Gorenstein terminal singularities which admit small resolution, 3.17. Next we consider deformations of models of trees. The approach is, roughly, as follows: we know how to deform equations of a small tree with four leaves and one inner edge, the result of the deformation is another tree with the inner edge “flopped”:



Applying the GIT quotient formula, 2.26, we are able to use this elementary deformation associated to four leaves trees to get a similar deformation for *every* inner edge of any tree, 3.24. This implies the result about deforming one geometric model to another, 3.26.

In the last part of section 3, we discuss Hilbert-Ehrhard polynomial of models (both polytopes and varieties) of trees. We define a relative version of the polynomial and then a product of such polynomials which is related to gluing respective trees. The elementary deformation procedure implies associativity of the product which not only implies the invariance of the Hilbert-Ehrhard polynomial for trees with the same number of leaves but also provides a simple formula for computing it, 3.38.

The appendix contains some computations. Firstly we prove that the polytopes of the 3-valent trees are normal which is needed to ensure the proper definition of their geometrical models. Next, using the [polymake] software we verify a simple (yet 9-dimensional) example to check that the polytope models of different trees in this case are different. The question if the polytope (or geometric) models of non-isomorphic trees are non-isomorphic is open, c.f. [Allman, Rhodes, '05]. Finally, we make numerical experiments (using [maxima] and [gnuplot]) to look at the behaviour of the relative volume distribution which measures the (normalized) volume of the model with respect to a fixed leaf of a tree.

The paper uses consistently the language of algebraic geometry, including toric geometry. We ignore, or barely mention, relations to algebraic statistic and biology, suggesting the reader to look into [Pachter, Sturmfels] (or into [ERSS] for a concise version of exposition), to get an idea about the background of the problems that we

deal with. It was our primary intention to make the present paper self-contained so that it can be read as it is by an algebraic geometer with no knowledge of its possible applications outside algebraic geometry. On the other hand, a reader who is not familiar with algebraic geometry but is interested in acquiring ideas which are important in our approach (regarding quotients and deformations) is advised to look into [Reid '92] and [Altman] for a short exposition to these matters.

We would like to thank Jarosław Buczyński for his remarks and Piotr Zwiernik for bringing this subject to our attention.

0.1 Notation

- $|\mathcal{A}|$ denotes cardinality of a finite set \mathcal{A} .
- A lattice is a finitely generated free abelian group.
- Depending on the context a subscript denotes the extension of the basic ring or a fiber of a morphism, e.g. $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.
- Given a finite dimensional vector space (or a lattice) V with a basis $\{v_1, \dots, v_n\}$, by $\{v_1^*, \dots, v_n^*\}$ we will denote the dual base of V^* , that is $v_i^*(v_i) = 1$ and $v_i^*(v_j) = 0$ if $i \neq j$.

1 Preliminaries: phylogenetic trees.

Summary: (for algebraic geometers) phylogenetic trees are a clever way of describing linear subsystems of Segre system on the product of projective spaces. In case of binary symmetric trees the question is to find subsystems of sections of Segre system on a product of \mathbb{P}^1 's invariant with respect to some $\mathbb{Z}_2^{|\mathcal{N}|}$ action.

1.1 Trees and linear algebra

Notation 1.1. A tree \mathcal{T} is a simply connected graph (1-dimensional CW complex) with a set of edges $\mathcal{E} = \mathcal{E}(\mathcal{T})$ and vertices $\mathcal{V} = \mathcal{V}(\mathcal{T})$ and the (unordered) boundary map $\partial : \mathcal{E} \rightarrow \mathcal{V}^{\wedge 2}$, where $\mathcal{V}^{\wedge 2}$ denotes the set of unordered pairs of distinct elements in \mathcal{V} . The number $|\mathcal{E}| \geq 1$ is, by definition, the number of edges of \mathcal{T} , then number of vertices $|\mathcal{V}|$ is $|\mathcal{E}| + 1$. We write $\partial(e) = \{\partial_1(e), \partial_2(e)\}$ and say v is a vertex of e , or e contains v if $v \in \{\partial_1(e), \partial_2(e)\}$, we simply write $v \in e$. The valency of a vertex v is the number of edges which contain v (the valency is positive since \mathcal{T} is connected

and we assume it has at least one edge). A vertex v is called a leaf if its valency is 1, otherwise it is called an inner vertex or a node. If the valency of each inner node is m then the tree will be called m -valent. The set of leaves and nodes will be denoted \mathcal{L} and \mathcal{N} , respectively, $\mathcal{V} = \mathcal{L} \cup \mathcal{N}$. An edge which contains a leaf is called a petiole, an edge which is not a petiole is called an inner edge (or branch), and the set of inner edges will be denoted by \mathcal{E}^o .

Example 1.2. An *caterpillar* of length n is a 3-valent tree with n inner edges and $n + 1$ inner nodes whose defoliation (i.e. after removing all leaves and petioles) is just a string of edges. That is, there are exactly two inner nodes to which of them there are attached two petioles (we call them heads or tails), any other inner node has exactly one petiole (called a leg) attached.



◦

Notation 1.3. Let W be a (complex, finite dimensional) vector space with a distinguished basis, sometimes called letters: $\{\alpha_0, \alpha_1, \alpha_2 \dots\}$. We consider the map $\sigma : W \rightarrow \mathbb{C}$, such that $\sigma(\alpha_i) = 1$ for every i , that is $\sigma = \sum \alpha_i^*$.

Let \widehat{W} be a subspace of the second tensor product $W \otimes W$. An element $\sum_{i,j} a_{ij}(\alpha_i \otimes \alpha_j)$ of \widehat{W} can be represented as a matrix (a_{ij}) . Through the present paper we will assume that these matrices are symmetric, that is \widehat{W} is contained in $S^2(W)$.

Given a tree \mathcal{T} and a vector space W , and a subspace $\widehat{W} \subset W \otimes W$ we associate to any vertex v of $\mathcal{V}(\mathcal{T})$ a copy of W denoted by W_v and for any edge $e \in \mathcal{E}(\mathcal{T})$ we associate a copy of \widehat{W} understood as the subspace in the tensor product $\widehat{W}^e \subset W_{\partial_1(e)} \otimes W_{\partial_2(e)}$. Note that although the pair $\{\partial_1(e), \partial_2(e)\}$ is unordered, this definition makes sense since \widehat{W} consists of symmetric tensors. Elements of \widehat{W}^e will be written as (symmetric) matrices $(a_{\alpha_i, \alpha_j}^e)$.

Definition 1.4. The triple $(\mathcal{T}, W, \widehat{W})$ together with the above association is called a (symmetric, unrooted) phylogenetic tree.

Construction 1.5. Let us consider a linear map of tensor products

$$\widehat{\Psi} : \widehat{W}^{\mathcal{E}} = \bigotimes_{e \in \mathcal{E}} \widehat{W}^e \longrightarrow W_{\mathcal{V}} = \bigotimes_{v \in \mathcal{V}} W_v$$

defined by setting its dual as follows

$$\widehat{\Psi}^*(\otimes_{v \in \mathcal{V}} \alpha_v^*) = \otimes_{e \in \mathcal{E}} (\alpha_{\partial_1(e)} \otimes \alpha_{\partial_2(e)})^*_{|\widehat{W}^e}$$

where α_v stands for an element of the chosen basis $\{\alpha_i\}$ of the space W_v . The complete affine geometric model of the phylogenetic tree $(\mathcal{T}, W, \widehat{W})$ is the image of the associated multi-linear map

$$\widetilde{\Psi} : \prod_{e \in \mathcal{E}} \widehat{W}^e \longrightarrow W_{\mathcal{V}} = \bigotimes_{v \in \mathcal{V}} W_v$$

The induced rational map of projective varieties will be denoted by Ψ :

$$\Psi : \prod_{e \in \mathcal{E}} \mathbb{P}(\widehat{W}^e) \dashrightarrow \mathbb{P}(W_{\mathcal{V}}) = \mathbb{P}(\bigotimes_{v \in \mathcal{V}} W_v)$$

and the closure of the image of Ψ is called the complete projective geometric model, or just the complete model of $(\mathcal{T}, W, \widehat{W})$. The maps $\widetilde{\Psi}$ and Ψ are called the parameterization of the respective model.

Given a set of vertices of the tree we can “hide” them by applying the map $\sigma = \sum_i \alpha_i^*$ to their tensor factors. In what follows will hide inner nodes and project to leaves. That is, we consider the map

$$\Pi_{\mathcal{L}} : W_{\mathcal{V}} = \bigotimes_{v \in \mathcal{V}} W_v \rightarrow W_{\mathcal{L}} = \bigotimes_{v \in \mathcal{L}} W_v$$

$$\Pi_{\mathcal{L}} = (\otimes_{v \in \mathcal{L}} id_{W_v}) \otimes (\otimes_{v \in \mathcal{N}} \sigma_{W_v})$$

Definition 1.6. *The affine geometrical model of a phylogenetic tree $(\mathcal{T}, W, \widehat{W})$ is an affine subvariety of $W_{\mathcal{L}} = \bigotimes_{v \in \mathcal{L}} W_v$ which is the image of the composition $\Phi = \Pi_{\mathcal{L}} \circ \Psi$. Respectively, the projective geometrical model, or just a model, denoted by $X(\mathcal{T})$ is the underlying projective variety in $\mathbb{P}(W_{\mathcal{L}})$. For $X = X(\mathcal{T})$ by $\mathcal{O}_X(1)$ we will denote the the hyperplane section bundle coming from the embedding in the projective space $\mathbb{P}(W_{\mathcal{L}})$.*

Note that $X(\mathcal{T})$ is the closure of the image of the respective rational map

$$\prod_{e \in \mathcal{E}} \mathbb{P}(\widehat{W}^e) \dashrightarrow \mathbb{P} \left(\bigotimes_{v \in \mathcal{L}} W_v \right)$$

which is defined by a special linear subsystem in the Segre linear system $|\bigotimes_{e \in \mathcal{E}} p_{\mathbb{P}(\widehat{W}^e)}^* \mathcal{O}_{\mathbb{P}(\widehat{W}^e)}(1)|$, where $p_{\mathbb{P}(\widehat{W}^e)}^*$ is the projection from the product to the respective component. We will call this map a *rational parametrization* of the model.

The above definition of parametrization is an unrooted and algebracized version of what is commonly considered in the literature, see e.g. [Allman, Rhodes '03], [Sturmfels, Sullivant] or [CGS].

1.2 Binary symmetric trees.

Depending on the choice of $\widehat{W} \subset W \otimes W$ we get different phylogenetic trees and their models. A natural assumption is that in the matrix representation the elements of \widehat{W} the sum of the numbers in each row and each column is the same (in applications, these numbers would stand for the probability distribution so their sum should be equal to 1). If W is of dimension 2 this is equivalent to saying that the respective matrix is of the form

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

for some a and b in \mathbb{C} .

From now on we will consider binary symmetric phylogenetic trees, that is, we assume that dimension of W is 2 and \widehat{W} consists of matrices (tensors) satisfying the above symmetric condition. The elements of the distinguished basis of W will be denoted α and β . Note that \widehat{W} has dimension 2 as well. We will call them binary symmetric trees or just trees when the context is obvious. Our task is to understand geometric models of these trees.

Example 1.7. Let \mathcal{T} be a tree which has one inner node v_0 , three leaves v_1, v_2, v_3 whose petioles we denote, respectively, by e_1, e_2, e_3 . We denote the basis of W_{v_i} by α_i, β_i , while \widehat{W}^{e_i} consists of matrices $\begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix}$. Then the parameterization map

$$\tilde{\Psi} : \widehat{W}^{e_1} \times \widehat{W}^{e_2} \times \widehat{W}^{e_3} \longrightarrow W_{v_1} \otimes W_{v_2} \otimes W_{v_3}$$

is as follows:

$$\begin{aligned} \tilde{\Psi}(a_1, b_1, a_2, b_2, a_3, b_3) = & \\ & (a_1 a_2 a_3 + b_1 b_2 b_3) \cdot (\alpha_1 \otimes \alpha_2 \otimes \alpha_3 + \beta_1 \otimes \beta_2 \otimes \beta_3) + \\ & (b_1 a_2 a_3 + a_1 b_2 b_3) \cdot (\beta_1 \otimes \alpha_2 \otimes \alpha_3 + \alpha_1 \otimes \beta_2 \otimes \beta_3) + \\ & (a_1 b_2 a_3 + b_1 a_2 b_3) \cdot (\alpha_1 \otimes \beta_2 \otimes \alpha_3 + \beta_1 \otimes \alpha_2 \otimes \beta_3) + \\ & (a_1 a_2 b_3 + b_1 b_2 a_3) \cdot (\alpha_1 \otimes \alpha_2 \otimes \beta_3 + \beta_1 \otimes \beta_2 \otimes \alpha_3) \end{aligned}$$

◦

Notation 1.8. Let $\rho : W \rightarrow W$ be a linear involution $\rho(\alpha) = \beta$, $\rho(\beta) = \alpha$, the map ρ is reflection with respect to the linear space W^ρ spanned by $\alpha + \beta$. We note that on \widehat{W} the right and left action of ρ coincide, i.e. $(\rho \otimes id_W)_{|\widehat{W}} = (id_W \otimes \rho)_{|\widehat{W}}$, and the resulting involution will be denoted by $\widehat{\rho}$, note that

$$\widehat{\rho} \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = \begin{bmatrix} b & a \\ a & b \end{bmatrix}$$

In particular, $\rho \otimes \rho$ is identity on \widehat{W} .

Given a binary symmetric tree $(\mathcal{T}, W, \widehat{W})$ we define respective involutions:

$$\rho_{\mathcal{V}} = \bigotimes_{v \in \mathcal{V}} \rho_v : \bigotimes_{v \in \mathcal{V}} W_v \rightarrow \bigotimes_{v \in \mathcal{V}} W_v$$

$$\rho_{\mathcal{L}} = \bigotimes_{v \in \mathcal{L}} \rho_v : \bigotimes_{v \in \mathcal{L}} W_v \rightarrow \bigotimes_{v \in \mathcal{L}} W_v$$

Let $W_{\mathcal{V}}^\rho = (\bigotimes_{v \in \mathcal{V}} W_v)^{\rho_{\mathcal{V}}}$ and $W_{\mathcal{L}}^\rho = (\bigotimes_{v \in \mathcal{L}} W_v)^{\rho_{\mathcal{L}}}$ be their fixed points, that is the maximal subspace on which $\rho_{\mathcal{V}}$ and, respectively, $\rho_{\mathcal{L}}$ acts trivially.

Lemma 1.9. *The image of Ψ is contained in $\mathbb{P}(W_{\mathcal{V}}^\rho)$ and the induced map*

$$\Psi : \prod_{e \in \mathcal{E}} \mathbb{P}(\widehat{W}^e) \rightarrow \mathbb{P}(W_{\mathcal{V}}^\rho)$$

is Segre embedding.

Proof. We want to prove that $\widehat{\Psi}$ maps $\widehat{W}^{\mathcal{E}}$ isomorphically to the space $W_{\mathcal{V}}^\rho$. First let us note that

$$\begin{aligned} \widehat{\Psi}^*(\rho_{\mathcal{V}}^*(\bigotimes_{v \in \mathcal{V}} \alpha_v^*)) &= \widehat{\Psi}^*(\bigotimes_{v \in \mathcal{V}} \rho_v^*(\alpha_v^*)) = \\ \bigotimes_{e \in \mathcal{E}} (\rho_{\partial_1(e)}^*(\alpha_{\partial_1(e)}^*) \otimes \rho_{\partial_2(e)}^*(\alpha_{\partial_2(e)}^*)) &= \bigotimes_{e \in \mathcal{E}} (\alpha_{\partial_1(e)}^* \otimes \alpha_{\partial_2(e)}^*) = \widehat{\Psi}^*(\bigotimes_{v \in \mathcal{V}} \alpha_v^*) \end{aligned}$$

(where, again, as in 1.5 α_v denotes either α or β in the space W_v) so that $\widehat{\Psi}^* \circ \rho_{\mathcal{V}}^* = \widehat{\Psi}^*$ which implies $\rho_{\mathcal{V}} \circ \widehat{\Psi} = \widehat{\Psi}$, hence $\text{im}(\widehat{\Psi}) \subset W_{\mathcal{V}}^\rho$.

Next, let us note that $\dim W_{\mathcal{V}}^\rho = 2^{|\mathcal{V}|-1}$ so that it is equal to $\dim \widehat{W}^{\mathcal{E}}$ because $|\mathcal{E}| = |\mathcal{V}| - 1$. The proof (e.g. by induction with respect to $|\mathcal{V}|$) is instantaneous if one observes that the basis of $W_{\mathcal{V}}$ can be made of tensor products of $(+1)$ and (-1) eigenvectors of each ρ_v and thus $W_{\mathcal{V}}$ splits into the sum of $(+1)$ and (-1) eigenspaces of $\rho_{\mathcal{V}}$, each of the same dimension.

Now, to conclude the proof we have to show that $\widehat{\Psi}$ is injective which is equivalent to $\widehat{\Psi}^*$ being surjective. Note that \widehat{W}^* is spanned by two forms:

$$\gamma_0 \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = a, \quad \gamma_1 \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = b$$

and $\gamma_1 = \gamma_0 \circ \widehat{\rho}$.

Now given an element $\otimes_{e \in \mathcal{E}} \gamma_{i(e)} \in \otimes_{e \in \mathcal{E}} \widehat{W}^e$ we define inductively a sequence α_v^* , indexed by vertices of \mathcal{T} such that $\widehat{\Psi}(\otimes_{v \in \mathcal{V}} \alpha_v^*) = \otimes_{e \in \mathcal{E}} \gamma_{i(e)}$. We choose a vertex v_0 and set $\alpha_{v_0}^*$ to be either α^* or β^* . Now suppose that α_v^* are defined for v in a subtree \mathcal{T}' of \mathcal{T} . Suppose that v' is not in \mathcal{T}' but is joined to a vertex v'' in \mathcal{T}' by and edge e' . Then we set $\alpha_{v'}^* = \alpha_{v''}^*$ if $\gamma_{i(e')} = \gamma_0$ or $\alpha_{v'}^* = \rho(\alpha_{v''}^*)$ if $\gamma_{i(e')} = \gamma_1$. \square

Notation 1.10. Let us choose a node $v \in \mathcal{N}$. We consider an involution $\widehat{\rho}_v^\mathcal{E}$ on the space $\widehat{W}^\mathcal{E} = \otimes_{e \in \mathcal{E}} \widehat{W}^e$. First, for any $e \in \mathcal{E}$ we set $\widehat{\rho}_v^\mathcal{E} = id_{\widehat{W}^e}$ if v is not a vertex of e and $\widehat{\rho}_v^\mathcal{E} = \widehat{\rho}$ if v is a vertex of e . Next, we define $\widehat{\rho}_v^\mathcal{E} = \otimes_{e \in \mathcal{E}} \widehat{\rho}_v^e$. Let $G_\mathcal{N}$ be the group of automorphisms of $\widehat{W}^\mathcal{E}$ generated by involutions $\widehat{\rho}_v^\mathcal{E}$, for $v \in \mathcal{N}$.

The following observation about a convenient choice of coordinates is sometimes referred to as a Fourier transform, see e.g. [Sturmfels, Sullivant].

Lemma 1.11. $G_\mathcal{N} \cong \mathbb{Z}_2^{|\mathcal{N}|}$, the action of $G_\mathcal{N}$ restricts to $\prod_{e \in \mathcal{E}} \widehat{W}^e$ and it is diagonalizable.

Proof. This follows immediately from the definition of the action of $\widehat{\rho}_v^\mathcal{E}$ on $\widehat{W}^\mathcal{E}$. Namely, in basis of \widehat{W}^e consisting of $a + b$ and $a - b$ the action of $\widehat{\rho}_v^\mathcal{E}$ is diagonal. \square

Lemma 1.12. The map $\widehat{\Psi}^* \circ \Pi_\mathcal{L}^*$ maps $(W_\mathcal{L}^\rho)^*$ injectively into the space $((\widehat{W}^\mathcal{E})^*)^{G_\mathcal{N}}$ on which $G_\mathcal{N}$ acts trivially.

Proof. The proof is similar to that of 1.9. Firstly, the map $\widehat{\Psi}^* \circ \Pi_\mathcal{L}^*$ is injective on $(W_\mathcal{L}^\rho)^*$ because of 1.9 and injectivity of $\Pi_\mathcal{L}^*$. Next, we note that the action of $G_\mathcal{N}$ is trivial on its image. Indeed, we define $\rho_v^\mathcal{V} = \otimes_{w \in \mathcal{V}} \rho_v^w$, where, for any $w \in \mathcal{V}$ we set $\widehat{\rho}_v^w = id_{W_w}$ if $w \neq v$ and $\widehat{\rho}_v^w = \rho$ if $w = v$. Then for $v \in \mathcal{N}$ we have

$$(\widehat{\rho}_v^\mathcal{E})^* \circ \widehat{\Psi}^* = \widehat{\Psi}^* \circ (\rho_v^\mathcal{V})^* \quad \text{and} \quad \Pi_\mathcal{L} \circ \rho_v^\mathcal{V} = \Pi_\mathcal{L}$$

where the first equality follows directly from the definition of the map $\widehat{\Psi}$, 1.5. This implies $(\widehat{\rho}_v^\mathcal{E})^* \circ \widehat{\Psi}^* \circ \Pi_\mathcal{L}^* = \widehat{\Psi}^* \circ \Pi_\mathcal{L}^*$ which is what we want. \square

We will prove that, in fact, $\widehat{\Psi}^* \circ \Pi_{\mathcal{L}}^*$ is an isomorphism, 2.12, so that the geometric model of the tree is defined by $G^{\mathcal{N}}$ invariant sections of the Segre linear system. Because of 1.11 $G^{\mathcal{N}}$ can be treated as a subgroup of a complex torus and thus we can use toric geometry.

2 Toric geometry.

Summary: We study invariants of an action of $\mathbb{Z}_2^{\mathcal{N}}$ on $(\mathbb{P}^1)^{\times |\mathcal{E}|}$ and a related polytope in the cube $[0, 1]^{|\mathcal{E}|}$ which we call a polytope model of the tree. The polytope models are used to define geometric models in terms of toric geometry. These polytopes turn out to be fiber products of elementary ones. This leads to interpreting the geometrical model of a tree as a quotient of products.

2.1 Lattice of a tree and the action of the torus

Given a tree \mathcal{T} we encode it in terms of dual lattices.

Definition 2.1. *Let \mathcal{T} be a tree with the set of vertices \mathcal{V} and the set of edges \mathcal{E} . We set $M = M(\mathcal{T}) = \bigoplus_{e \in \mathcal{E}} \mathbb{Z} \cdot e$ to be a lattice, or free abelian group, spanned on the set \mathcal{E} . Let $N = N(\mathcal{T}) = \text{Hom}(M, \mathbb{Z})$ be the dual lattice. We represent elements of \mathcal{V} as elements of N . Namely, for $v \in \mathcal{V}$ we set $v(e) = 1$ if e contains the vertex v and $v(e) = 0$ otherwise. The pair (M, N) together with the choice of the basis \mathcal{E} of M and set $\mathcal{V} \subset N$ is called the lattice pair of the tree \mathcal{T} .*

From this point on we identify the edges and the vertices of \mathcal{T} with the respective elements in $M(\mathcal{T})$ and $N(\mathcal{T})$. The elements of the basis of N dual to $\{e \in \mathcal{E}\}$ will be denoted by e^* . Then for any $v \in \mathcal{V}$ we have, by definition, $v = \sum_{e \ni v} e^* : N \rightarrow \mathbb{Z}$. In particular, v is a leaf if and only if $v = e^*$ for some e , which is a petiole for v .

Let us recall that $|\mathcal{V}| = |\mathcal{E}| + 1$ so the set of vertices has to be linearly dependent in N . The set of vertices of \mathcal{T} can be divided into two disjoint classes, say $\mathcal{V} = \mathcal{V}^- \cup \mathcal{V}^+$, each class consisting of vertices which can be reached one from another by passing through an even number of edges.

Lemma 2.2. *The equality $\sum_{v \in \mathcal{V}^-} v = \sum_{v \in \mathcal{V}^+} v$ is, up to multiplication by a constant, the only linear relation in N between vectors v from \mathcal{V} . In particular, any proper subset of \mathcal{V} consists of linearly independent vectors in N .*

Proof. Suppose that $\sum a_v \cdot v = 0$, for some $a_v \in k$. For any $e \in \mathcal{E}$ we have

$$(\sum a_v \cdot v)(e) = a_{\partial_1(e)} + a_{\partial_2(e)}$$

and therefore $a_{\partial_1(e)} = -a_{\partial_2(e)}$. Thus we get the desired relation. \square

The operations on trees can be translated to lattices, here is an example.

Construction 2.3. Let v_0 be a 2-valent inner node of \mathcal{T} which belongs to exactly two edges e_1 and e_2 . Let \mathcal{T}_{v_0} be a tree obtained by removing the node v_0 from \mathcal{T} and replacing the edges e_1 and e_2 by a single edge e_0 . Let (M, N) be the lattice pair of \mathcal{T} . We set $M_{v_0} \subset M$ to be the kernel of $e_2^* - e_1^*$ and $N_{v_0} = N/\mathbb{Z} \cdot (e_2^* - e_1^*)$, clearly M_{v_0} and N_{v_0} are dual. We define $e'_0 = e_1 + e_2$. Note that $\mathcal{E}_{v_0} = \mathcal{E} \setminus \{e_1, e_2\} \cup \{e'_0\}$ is a basis of M_{v_0} . For $v \in \mathcal{V} \setminus \{v_0\}$ by v' we denote the image of a vertex v under the projection $N \rightarrow N_{v_0}$ and set of all v' we denote by \mathcal{V}_{v_0} . One can verify easily the following.

Lemma 2.4. *The above defined pair (M_{v_0}, N_{v_0}) together with the above choice of \mathcal{E}_{v_0} and \mathcal{V}_{v_0} is the lattice pair of the tree \mathcal{T}_{v_0} obtained from \mathcal{T} by removing the 2-valent inner node v_0 .*

Now we set up the toric environment.

Construction 2.5. We deal with a binary symmetric tree $(\mathcal{T}, W, \widehat{W})$. Because of 1.11 for any edge $e \in \mathcal{E}$ there exists an inhomogeneous coordinate z_e on $\mathbb{P}_e^1 = \mathbb{P}(\widehat{W}^e)$ such that for $v \in e$ the action of $\rho_v^\mathcal{E}$ is as follows $\rho_v^\mathcal{E}(z_e) = -z_e$.

Let $T \cong \prod_{e \in \mathcal{E}} \mathbb{C}^*$ be a torus with coordinates $\{z_e \in \mathbb{C}^* : e \in \mathcal{E}\}$ and with the natural action

$$T \times \prod_{e \in \mathcal{E}} \mathbb{P}_e^1 \longrightarrow \prod_{e \in \mathcal{E}} \mathbb{P}_e^1$$

which is the multiplication of z_e 's coordinate-wise. We consider an injective map $\iota : G_{\mathcal{N}} \rightarrow T \cong \prod_{e \in \mathcal{E}} \mathbb{C}^*$ which is defined as follows. For any $\rho_v^\mathcal{E} \in G_{\mathcal{N}}$ we take $\iota(\rho_v^\mathcal{E}) \in T$ such that $z_e(\iota(\rho_v^\mathcal{E})) = -1$ if $v \in e$ and $z_e(\iota(\rho_v^\mathcal{E})) = 1$ if $v \notin e$. Then ι extends to a homomorphism of groups $\iota : G_{\mathcal{N}} \rightarrow T \cong \prod_{e \in \mathcal{E}} \mathbb{C}^*$ and the action of $G_{\mathcal{N}}$ on $\prod_{e \in \mathcal{E}} \mathbb{P}_e^1$ factors through ι .

We explain this situation using the lattices of the tree \mathcal{T} and toric geometry formalism. Our notation is consistent with this of standard toric geometry textbooks, e.g. [Oda] or [Fulton]. We take the torus $T = T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ with coordinates $z_e = \chi^e$, where $e \in \mathcal{E} \subset M$ is the distinguished basis. Recall that the elements of M can be identified with monomials in coordinates z_e , that is, each $u \in M$ such that $u = \sum a_i e_i$ represents a monomial $\chi^u = \prod z_{e_i}^{a_i}$. For $w \in N$ and $t \in \mathbb{C}^*$ the z_e -th coordinate of the respective point on $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is as follows $z_e(w \otimes t) = t^{w(e)}$. Moreover, recall that every element w of N can be identified with algebraic 1-parameter

subgroups λ_w of T_N . That is, for $w \in N$ and $t \in \mathbb{C}^*$ we set $\lambda_w(t)(z_e) = t^{w(e)} \cdot z_e$. In short, $N = \text{Hom}_{\text{alg}}(\mathbb{C}^*, T_N)$ and $M = \text{Hom}_{\text{alg}}(T_N, \mathbb{C}^*)$, [Fulton, Sect 2.3]

The complexified lattice $N_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}$ can be interpreted as the tangent space to the unit element in the torus T_N and we have the natural exponential map $N_{\mathbb{C}} \rightarrow T_N$. In particular, $z_e(\exp(2\pi i(w))) = \exp(2\pi i(w(e)))$. The image of the real vector space $N_{\mathbb{R}} \subset N_{\mathbb{C}}$ under this exponential map is the maximal compact real subgroup $\prod S^1$ of T_N . Using the exponential map we can relate the vertices $v \in \mathcal{N}$ viewed as elements of the lattice N to their respective automorphisms $\rho_v^{\mathcal{E}} \in G^{\mathcal{N}}$. The following lemma is not used directly in our arguments so we skip its proof.

Lemma 2.6. *For every $v \in \mathcal{V}$ we have $\exp(2\pi i(v/2)) = \iota(\rho_v^{\mathcal{E}})$. If \widehat{N} is the lattice spanned in $N_{\mathbb{R}}$ by N and $\mathcal{N}/2 = \{v/2 : v \in \mathcal{N}\}$ then the inclusion $N \hookrightarrow \widehat{N}$ yields an exact sequence of groups*

$$0 \longrightarrow \iota(G^{\mathcal{N}}) \longrightarrow T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \longrightarrow \widehat{N} \otimes_{\mathbb{Z}} \mathbb{C}^* \longrightarrow 0$$

For our purposes we need the following lemma which provides a clear description of functions on the torus T_N which are invariant with respect to the action of G_N .

Lemma 2.7. *A monomial function χ^u on T_N with $u \in M$ is invariant with respect to the action of $\rho_v^{\mathcal{E}}$ if and only if $v(u) \in 2\mathbb{Z}$.*

Proof. First, we note that, by definition, $\rho_v^{\mathcal{E}}(\chi^e) = \rho_v^{\mathcal{E}}(z_e) = (-1)^{v(e)} \cdot z_e = (-1)^{v(e)} \chi^e$. Next, we write the exponent of the monomial χ^u in terms of the distinguished coordinates: $u = \sum_{e \in \mathcal{E}} e^*(u)e$. Then, since $\rho_v^{\mathcal{E}}$ is a homomorphism, we get

$$\rho_v^{\mathcal{E}}(\chi^u) = (-1)^{\sum_e e^*(u) \cdot v(e)} \cdot \chi^u = (-1)^{v(u)} \cdot \chi^u$$

which concludes the proof. \square

Definition 2.8. *Given a tree \mathcal{T} with the lattice pair $(M, N) = (M(\mathcal{T}), N(\mathcal{T}))$ we define its normalized lattice pair $(\widehat{M}, \widehat{N}) = (\widehat{M}(\mathcal{T}), \widehat{N}(\mathcal{T}))$ as follows: $\widehat{M} = \{u \in M : \forall v \in \mathcal{V} \ v(u) \in 2\mathbb{Z}\}$ and \widehat{N} is a dual of \widehat{M} which contains N and the set $\mathcal{N}/2 = \{v/2 : v \in \mathcal{N}\}$.*

In view of 2.7 the lattice \widehat{M} contains monomials which are $G^{\mathcal{N}}$ invariant.

2.2 Polytope model of a tree

The complete Segre linear system on $\prod_{e \in \mathcal{E}} \mathbb{P}_e^1$ is spanned on monomials $\prod_{e \in \mathcal{E}} z_e^{\epsilon_e}$ where $\epsilon_e \in \{0, 1\}$. Equivalently, once the big torus action T_N on $\prod \mathbb{P}_e^1$ is chosen, the complete Segre system is represented by vertices of the unit cube $\square_M = \{u \in M_{\mathbb{R}} : \forall_i 0 \leq e_i^*(u) \leq 1\}$ in the space of characters $M_{\mathbb{R}}$, or by zero-one sequences indexed by \mathcal{E} .

Because of 1.9 and 1.12 we are interested in subsystems of the Segre linear systems or, equivalently, subsets of vertices of \square_M . If Δ is a polytope in $M_{\mathbb{R}}$ whose vertices are contained in the set of vertices of \square_M then we call it a subcube.

Definition 2.9. *Given a binary tree \mathcal{T} with its lattice pair (M, N) its polytope model $\Delta(\mathcal{T})$ is a polytope in the lattice M which is the convex hull of $\{u = \sum a_i e_i \in M : a_i = 0, 1 \text{ and } v(u) \in 2\mathbb{Z} \text{ for every } v \in \mathcal{N}\}$.*

We note that the vertices of Δ are precisely these among vertices of \square_M which are in the sublattice $\widehat{M} \subset M$ and because of 2.7 they are exactly these monomials in the complete Segre system which are invariant with respect to the action of $G_{\mathcal{N}}$.

Since the cube \square_M is the fundamental domain in dividing M modulo 2 we can interpret the elements of the complete Segre system as points in the linear space $M_{\mathbb{Z}_2} = M \otimes_{\mathbb{Z}} \mathbb{Z}_2$.

Lemma 2.10. *If the vertices of the cube \square_M are identified with the points in linear space $M \otimes_{\mathbb{Z}_2}$ then vertices of $\Delta(\mathcal{T})$ form the linear subspace $\mathcal{N}^{\perp} \subset M \otimes \mathbb{Z}_2$ of zeros of forms $v \in N \otimes \mathbb{Z}_2$, where $v \in \mathcal{N}(\mathcal{T})$.*

Proof. This is a restatement of 2.7. □

Corollary 2.11. *The polytope $\Delta(\mathcal{T})$ has $2^{|\mathcal{L}|-1}$ vertices.*

Proof. We use 2.10: by 2.2 the elements v 's are linearly independent in $N \otimes \mathbb{Z}_2$ so dimension of the space of their zeroes in $M \otimes \mathbb{Z}_2$ is $|\mathcal{E}| - |\mathcal{N}| = |\mathcal{L}| - 1$. □

Using the above information we can conclude identifying the linear subsystem in the Segre system which defines the projective model of a binary symmetric tree.

Theorem 2.12. *In the situation of section 1.2 the map $\widehat{\Psi}^* \circ \Pi_{\mathcal{L}}^*$ maps $(W_{\mathcal{L}}^{\rho})^*$ isomorphically to $(\widehat{W}^{\mathcal{E}})^{*})^{G_{\mathcal{N}}}$. In particular, in terms of the toric coordinates on $\prod_{e \in \mathcal{E}} \mathbb{P}(\widehat{W}^e)$ introduced in section 2.1, the rational parametrization map*

$$\prod_{e \in \mathcal{E}} \mathbb{P}(\widehat{W}^e) \rightarrow \mathbb{P}(W_{\mathcal{L}}^{\rho}) \subset \mathbb{P}\left(\bigotimes_{v \in \mathcal{L}} W_v\right)$$

is defined by elements of the Segre linear system on $\prod_{e \in \mathcal{E}} \mathbb{P}(\widehat{W}^e)$ which are associated to vertices of $\Delta(\mathcal{T})$.

Proof. By the construction the vertices of $\Delta(\mathcal{T})$ are these monomial in the Segre system which are invariant with respect to the action of $G^{\mathcal{N}}$. In other words they form a basis for $((\widehat{W}^{\mathcal{E}})^*)^{G^{\mathcal{N}}}$. In 1.12 we have proved that the parametrization map injects $(W_{\mathcal{L}}^{\rho})^*$ into the space $((\widehat{W}^{\mathcal{E}})^*)^{G^{\mathcal{N}}}$ and now by 2.11 they are of the same dimension so this is an isomorphism. \square

Thus we have determined that studying projective geometric models of binary symmetric trees is essentially equivalent to understanding their polytopes. We start with the simplest, in fact trivial, example.

Example 2.13. Let \mathcal{T} be a tree consisting of two leaves, two petioles e_1 and e_2 , and one inner node v_0 . Then $\Delta(\mathcal{T})$ is spanned on 0 and $e_1 + e_2$.

◦

More generally we have the following result which extends 2.4.

Lemma 2.14. Suppose that \mathcal{T} is a tree with a 2-valent node v_0 . Let \mathcal{T}_{v_0} be a tree obtained from \mathcal{T} by removing v_0 , as in the situation of lemma 2.4. Then, under the natural inclusion $M(\mathcal{T}_{v_0}) \subset M(\mathcal{T})$ we have $\Delta(\mathcal{T}_{v_0}) = \Delta(\mathcal{T})$ and $\widehat{M}(\mathcal{T}_{v_0}) = \widehat{M}(\mathcal{T}) \cap M(\mathcal{T}_{v_0})_{\mathbb{R}}$

Proof. We use the notation of 2.4, in particular e_1 and e_2 denote the edges containing v_0 and $M(\mathcal{T}_{v_0}) = M_{v_0} = \ker(e_2^* - e_1^*)$. Note that the parity of the node $v_0 = e_1^* + e_2^*$ is equivalent to that of $e_2^* - e_1^*$, in particular for $u \in M(\mathcal{T}_{v_0}) = \ker(e_2^* - e_1^*)$ we have $v_0(u) \in 2\mathbb{Z}$. Since $\mathcal{N}(\mathcal{T}) = \mathcal{N}(\mathcal{T}_{v_0}) \cup \{v_0\}$ the conditions defined by $\mathcal{N}(\mathcal{T})$ and $\mathcal{N}(\mathcal{T}_{v_0})$ on $M(\mathcal{T}_{v_0})_{\mathbb{R}}$ are the same. Similarly, the conditions defining $\Delta(\mathcal{T})$ and $\Delta(\mathcal{T}_{v_0})$ inside $M_{v_0} \otimes \mathbb{R} \cap \square_M = \square_{M_{v_0}}$ are the same. \square

By 2.23 we have that removing the 2-valent node does not change the model of the tree. Thus, from now on we consider trees with no 2-valent nodes.

A star tree is a tree which has exactly one inner node, a star tree with d leaves will be denoted by \mathcal{T}^{*d} .

Lemma 2.15. If $d \geq 3$ then vertices of $\Delta(\mathcal{T}^{*d})$ generate $\widehat{M}(\mathcal{T}^{*d})$, in particular $\dim \Delta(\mathcal{T}^{*d}) = d$.

Proof. If $\{e_i\}$ is the set of edges then $\Delta(\mathcal{T}^{*d})$ contains sums $e_i + e_j$ for all possible pairs $i \neq j$ and for $d \geq 3$ they span $\widehat{M}(\mathcal{T}^{*d})$ which is of index 2 in $M(\mathcal{T}^{*d})$. \square

Example 2.16. The vertices of the polytope $\Delta(\mathcal{T}^{*3})$ with edges e_0, e_1, e_2 are as follows: $0, e_1 + e_2, e_2 + e_0$ and $e_0 + e_1$ so that $\Delta(\mathcal{T}^{*3})$ is a 3-dimensional tetrahedron. If $\widehat{M} \subset M$ is the sublattice spanned by the vertices of $\Delta(\mathcal{T}^{*3})$ then $M/\widehat{M} \cong \mathbb{Z}_2$.

The inequalities defining $\Delta(\mathcal{T}^{*3})$ are as follows

$$(-v/2)(\cdot) \geq -1, \quad (v/2 - e_0^*)(\cdot) \geq 0, \quad (v/2 - e_1^*)(\cdot) \geq 0, \quad (v/2 - e_2^*)(\cdot) \geq 0.$$

where, recall, $v = e_0^* + e_1^* + e_2^*$.

◦

Construction 2.17. A *pointed tree* (\mathcal{T}, ℓ) is a pair consisting of a tree \mathcal{T} and a leaf $\ell \in \mathcal{L}(\mathcal{T})$. Given two pointed trees (\mathcal{T}_1, ℓ_1) and (\mathcal{T}_2, ℓ_2) we define their graft as follows: $\mathcal{T} = \mathcal{T}_1 \ell_1 \vee_{\ell_2} \mathcal{T}_2$ is a tree obtained by removing from each \mathcal{T}_i the leaf ℓ_i and identifying their respective petioles which becomes an inner edge of the resulting tree \mathcal{T} .

For example, a graft of two trees of type \mathcal{T}^{*3} with distinguished leaves denoted by \circ is the following operation

$$\diagup \circ \vee \circ \diagdown = \diagup \diagdown$$

Let us take two lattices M_1 and M_2 with distinguished bases $\{e_0^i, \dots, e_{m_i}^i\}$, for $i = 1, 2$ and respective subcube polytopes Δ_i , each of them having the set of vertices \mathcal{A}_i . Let $\ell_i = (e_0^i)^* : M_i \rightarrow \mathbb{Z}$ be the projection to the zeroth coordinate and, by abuse of notation, by the same letter we will denote its composition with the projection $M_1 \times M_2 \rightarrow M_i \rightarrow \mathbb{Z}$. Now we can take fiber product of each of these objects, relative over the projection, e.g. $M_1 \ell_1 \times_{\ell_2} M_2 \subset M_1 \times M_2$ consists of pairs (u_1, u_2) such that $\ell_1(u_1) = \ell_2(u_2)$. In other words $M_1 \ell_1 \times_{\ell_2} M_2 = \ker(\ell_1 - \ell_2)$ and $\Delta_1 \ell_1 \times_{\ell_2} \Delta_2 = (\Delta_1 \times \Delta_2) \cap \ker(\ell_1 - \ell_2)$

Lemma 2.18. *In the above situation $\Delta = \Delta_1 \ell_1 \times_{\ell_2} \Delta_2$ is a subcube polytope in $M = M_1 \ell_1 \times_{\ell_2} M_2$ with the set of vertices $\mathcal{A} = \mathcal{A}_1 \ell_1 \times_{\ell_2} \mathcal{A}_2$. In general, if $\Delta_i \subset (M_i)_{\mathbb{R}}$ and $\ell_i : M_i \rightarrow \mathbb{Z}$ are lattice homomorphisms such that $\ell_i(\Delta_i) \subset [0, 1]$ then the set of vertices of $\Delta = \Delta_1 \ell_1 \times_{\ell_2} \Delta_2$ is the fiber product of the vertices of Δ_i 's.*

Proof. The only non-trivial thing is to show that all vertices of Δ are in the fiber product of vertices of Δ_1 and Δ_2 . Since $\Delta = (\Delta_1 \times \Delta_2) \cap \ker(\ell_1 - \ell_2)$ is a codimension 1 linear section of $\Delta_1 \times \Delta_2$ its vertices are either vertices of $\Delta_1 \times \Delta_2$ (which is what we want) or are obtained by intersecting the hyperplane $\ker(\ell_1 - \ell_2)$ with an edge of $\Delta_1 \times \Delta_2$. To this end, let us take two pairs of vertices $(u_1^1, u_2^1), (u_1^2, u_2^2)$, where u_j^i is a

vertex of Δ_j . Suppose that for some $t \in (0, 1)$ the point $u = t(u_1^1, u_2^1) + (1-t)(u_1^2, u_2^2)$ is in $\ker(\ell_1 - \ell_2)$, that is, we have

$$\ell_1(tu_1^1 + (1-t)u_1^2) = \ell_2(tu_2^1 + (1-t)u_2^2)$$

and moreover $\ell_1(u_1^1) \neq \ell_2(u_2^1)$. Thus, we may assume that $\ell_1(u_1^1) = 0$ and $\ell_2(u_2^1) = 1$. Hence, because of the above equality and since $t \neq 0, 1$, we get $\ell_1(u_1^2) = 1$, $\ell_2(u_2^2) = 0$ and $t = 1/2$. So $u = t(u_1^1, u_2^1) + (1-t)(u_1^2, u_2^2) = t(u_1^1, u_2^2) + (1-t)(u_1^2, u_1^1)$ and (u_1^1, u_2^2) , (u_1^2, u_1^1) are vertices of Δ so u lies in the interior of an edge of Δ . \square

Example 2.19. Let us consider two copies of a tetrahedron, as in example 2.16. That is, for $i = 1, 2$, in a lattice $M^i = \mathbb{Z}e_0^i \oplus \mathbb{Z}e_1^i \oplus \mathbb{Z}e_2^i$, we consider a tetrahedron Δ_i^3 spanned on vertices 0 , $e_0^i + e_1^i$, $e_1^i + e_2^i$ and $e_2^i + e_0^i$. We take the projections $(e_0^i)^* : M^i \rightarrow \mathbb{Z}$ and in the fiber product

$$M = M_1 \times_{\ell_1 \times \ell_2} M_2 = \ker((e_0^2 - e_0^1)^*) \subset M_1 \times M_2$$

by e_0 we denote the element $e_0^1 + e_0^2$. The resulting fiber product of tetrahedra

$$\Delta = (\Delta_1^3 \times \Delta_2^3) \cap M_{\mathbb{R}}$$

has the following vertices: $0, e_1^1 + e_2^1, e_1^2 + e_2^2, e_1^1 + e_2^1 + e_1^2 + e_2^2, e_0 + e_1^1 + e_2^1, e_0 + e_1^1 + e_2^2, e_0 + e_2^1 + e_2^2, e_0 + e_1^2 + e_2^2$.

◦

Proposition 2.20. Let (\mathcal{T}_1, ℓ_1) and (\mathcal{T}_2, ℓ_2) be two pointed trees. Then

$$\widehat{M}(\mathcal{T}_1 \times_{\ell_1 \vee \ell_2} \mathcal{T}_2) = \widehat{M}(\mathcal{T}_1) \times_{\ell_1 \times \ell_2} \widehat{M}(\mathcal{T}_2)$$

$$\Delta(\mathcal{T}_1 \times_{\ell_1 \vee \ell_2} \mathcal{T}_2) = \Delta(\mathcal{T}_1) \times_{\ell_1 \times \ell_2} \Delta(\mathcal{T}_2)$$

Proof. Let $M_1 = M(\mathcal{T}_1)$, $M_2 = M(\mathcal{T}_2)$, and similarly for N 's, \widehat{M} ' and \widehat{N} 's. We set $\mathcal{T} = \mathcal{T}_1 \times_{\ell_1 \vee \ell_2} \mathcal{T}_2$. Then, by construction 2.17, $M = M(\mathcal{T}) = M_1 \times_{\ell_1 \times \ell_2} M_2$ and $\square_M = \square_{M_1 \times_{\ell_1 \times \ell_2} M_2}$. The two projections $p_i : M \rightarrow M_i$ yield respective injections of $\text{Hom}(\cdot, \mathbb{Z})$ -spaces: $\iota_i : N_i \hookrightarrow N$ (in fact $N = (N_1 \times N_2)/\mathbb{Z}(\ell_1 - \ell_2)$). If \mathcal{N}_i and \mathcal{N} denote, respectively, inner nodes of \mathcal{T}_i and \mathcal{T} then $\mathcal{N} = \iota_1(\mathcal{N}_1) \cup \iota_2(\mathcal{N}_2)$. Since \widehat{N} , \widehat{N}_1 , \widehat{N}_2 are defined by extending N , N_1 , N_2 by $\mathcal{N}/2$, $\mathcal{N}_1/2$ and $\mathcal{N}_2/2$, respectively, it follows that $\widehat{N} = \widehat{N}_1 + \widehat{N}_2$ in $N_{\mathbb{R}}$. This implies the first equality of the lemma. Similarly, since the set \mathcal{N} determines vertices of \square_M which span $\Delta(\mathcal{T})$, see 2.9, we get the second equality. \square

The above result can be expressed as follows: the polygon of a tree \mathcal{T} is a fiber product of polygons of star trees associated to inner nodes of the tree, fibered over the relations encoded in the inner branches of the tree. Since $\Delta(\mathcal{T}^{*3})$ is a 3-dimensional tetrahedron this is especially straightforward in case of 3-valent trees.

For any inner node $v \in \mathcal{N}$ of a 3-valent tree \mathcal{T} we consider the lattice $M_v = \mathbb{Z}e_1^v \oplus \mathbb{Z}e_2^v \oplus \mathbb{Z}e_3^v$, where e_1, e_2 and e_3 are the three edges stemming from v . Inside M_v we have the tetrahedron Δ_v with vertices $0, e_1^v + e_2^v, e_2^v + e_3^v$ and $e_3^v + e_1^v$. We consider the big lattice $\widetilde{M} = \bigoplus_{v \in \mathcal{N}} M_v$ and $\widetilde{M}_{\mathbb{R}}$ contains the product $\prod_{v \in \mathcal{N}} \Delta_v$. Now for each inner edge e we have a form $\tilde{e}^* : \widetilde{M} \rightarrow \mathbb{Z}$ such that $\tilde{e}^*(e_i^v) = (-1)^\epsilon$ if $\partial_\epsilon(e) = v$ and $e_i^v = e$, and $\tilde{e}^*(e_i^v) = 0$ otherwise. Then the intersection $\bigcap_{e \in \mathcal{E}^o} \ker(\tilde{e}^*)$ can be identified with the lattice M , that is, we map $e \in \mathcal{E}^o$ to $e_i^v + e_{i'}^{v'}$ where $\partial(e) = (v, v')$, and $e_i^v = e_{i'}^{v'} = e$, while a petiole e is just mapped to its unique representation in \widetilde{M} . Then, by 2.20 we get

$$\Delta(\mathcal{T}) = \left(\prod_{v \in \mathcal{N}} \Delta_v \right) \cap \left(\bigcap_{e \in \mathcal{E}^o} \ker(\tilde{e}^*)_{\mathbb{R}} \right)$$

2.3 Geometric model of a tree

First, let us recall the construction of a projective tori variety from a lattice polytope of characters. Let \widehat{M} and \widehat{N} be dual lattices of characters and 1 parameter subgroups for a torus $T_{\widehat{N}} = \widehat{N} \otimes_{\mathbb{Z}} \mathbb{C}^*$.

Definition 2.21. A lattice polytope $\Delta \subset \widehat{M}_{\mathbb{R}}$ is called normal if

- the sublattice of \widehat{M} spanned by the differences of points in $\Delta \cap \widehat{M}$ is equal to \widehat{M}
- for every integer $d \geq 0$ any point in $d\Delta \cap \widehat{M}$ is equal to a sum of d points in $\Delta \cap \widehat{M}$.

Equivalently, the second condition in the above definition can be restated as follows. Let $\widehat{M}' = \widehat{M} \oplus \mathbb{Z}$ and take an affine map $i_1 : \widehat{M} \rightarrow \widehat{M}'$ such that $i_1(u) = (u, 1)$. Then Δ is normal in \widehat{M} if and only if the semigroup spanned in \widehat{M}' by $i_1(\Delta \cap \widehat{M})$ is equal to the semigroup of lattice points in cone spanned in $\widehat{M}'_{\mathbb{R}}$ by $i_1(\Delta)$, that is the semigroup $\mathbb{R}_{\geq 0}(i_1(\Delta)) \cap \widehat{M}'$.

Definition 2.22. Suppose that Δ is a normal polytope in \widehat{M} . Let A_{Δ}^d be a \mathbb{C} -linear space with the basis $\{\chi^u : u \in d\Delta \cap \widehat{M}\}$. We consider a graded \mathbb{C} -algebra $A(\Delta) = \bigoplus_{d \geq 0} A_{\Delta}^d$, with multiplication $\chi^{u_1} \chi^{u_2} = \chi^{u_1 + u_2}$. Then $X(\Delta) = \text{Proj } A(\Delta)$ is called the projective model of Δ .

We note that in the above situation $A(\Delta)$ is a normal ring, that is, it integrally closed in its field of fractions. This, by definition, is equivalent to saying that affine spectrum $\text{Spec}(A(\Delta))$ is a normal affine variety. In fact, in such a case $A(\Delta)$ is the semigroup algebra of $\mathbb{R}_{\geq 0}(i_1(\Delta)) \cap \widehat{M}'$ so $\text{Spec}(A(\Delta))$ is an affine toric variety with the big torus $T_{\widehat{N} \oplus \mathbb{Z}}$. In the projective case we have the following general result which summarizes properties of the projective model of a normal polytope, see [Oda, Sect. 2.1–2.4], [Sturmfels] or [Fulton].

Proposition 2.23. *Suppose that Δ is a normal polytope in the lattice \widehat{M} of characters of a torus $T_{\widehat{N}}$. Then the following holds:*

1. $X(\Delta)$ is a toric variety on which $T_{\widehat{N}}$ acts effectively,
2. $X(\Delta)$ is embedded in $\mathbb{P}^{|\Delta \cap \widehat{M}|-1}$ as a projectively normal variety such that $H^0(X(\Delta), \mathcal{O}_X(d)) = A_{\Delta}^d$,
3. Characters from $\Delta \cap \widehat{M}$ define a diagonal action of $T_{\widehat{N}}$ on $\mathbb{P}^{|\Delta \cap \widehat{M}|-1}$ which restricts to the torus action on $X(\Delta)$,
4. The induced action of $T_{\widehat{M}}$ on $H^0(X(\Delta), \mathcal{O}_X(d))$ is linearizable with weights in $d\Delta \cap \widehat{M}$.
5. $X(\Delta) \subset \mathbb{P}^{|\Delta \cap \widehat{M}|-1}$ is the closure of the image of the map $T_{\widehat{M}} \rightarrow \mathbb{P}^{|\Delta \cap \widehat{M}|-1}$ defined by the characters from $\Delta \cap \widehat{M}$

Because of A.5 the polytope model $\Delta(\mathcal{T})$ of a **3-valent** tree \mathcal{T} is normal so we can consider its projective model. The following is the key result of the paper which allows us to study projective models of binary symmetric trees in purely toric way.

Theorem 2.24. *Let $(\mathcal{T}, W, \widehat{W})$ be a binary symmetric **3-valent** tree. Then the varieties $X(\mathcal{T})$ and $X(\Delta(\mathcal{T}))$ are projectively equivalent in $\mathbb{P}(W_{\mathcal{L}}^{\rho}) = \mathbb{P}^{2^{|\mathcal{L}|-1}-1}$.*

Proof. By 2.12 the parametrization of $X(\mathcal{T})$ is defined as a rational map from $\prod_{e \in \mathcal{E}} \mathbb{P}(\widehat{W}^e)$ defined by characters of torus T_N which are vertices of $\Delta(\mathcal{T})$. Thus, $X(\mathcal{T})$ is the closure of the respective map $T_N \rightarrow \mathbb{P}^{2^{|\mathcal{L}|-1}-1}$. Since $\widehat{M} \subset M$ is the sublattice spanned by vertices of $\Delta(\mathcal{T})$ this factors to the map $T_{\widehat{N}} \rightarrow \mathbb{P}^{2^{|\mathcal{L}|-1}-1}$ the image of which defines $X(\Delta)$, 2.23. \square

2.4 1-parameter group action, quotients.

In this section we consider quotients of projective varieties as in Mumford's GIT [Mumford]. For a comprehensive exposition of the theory, including a relevant definition of good quotient we refer to [Białynicki-Birula]. In the present section as well as in section 3.3 we consider an algebraic action of a torus T on a projective variety $X \hookrightarrow \mathbb{P}^m$ which is given by a choice of weights hence it extends to the affine cone over X and thus it determines its linearization, its set of semi-stable points X^{ss} and its good quotient $X^{ss} \rightarrow X^{ss}/T$, see [Białynicki-Birula, Ch.6].

Construction 2.25. Let $\Delta_i \subset (\widehat{M}_i)_{\mathbb{R}}$, for $i = 1, 2$ be two lattice polytopes admitting unimodular covers hence normal, see A.1, and $X(\Delta_i) \subset \mathbb{P}^{n_i-1}$, where $n_i = |\widehat{M}_i \cap \Delta_i|$, their associated toric varieties. In $M^\times = \widehat{M}_1 \times \widehat{M}_2$ we take the product polytope $\Delta^\times = \Delta_1 \times \Delta_2$ which is also normal, reproduct-unimodular. Then the associated toric variety $X^\times = X(\Delta^\times) \subset \mathbb{P}^{n_1 n_2 - 1}$ is the Segre image of $X(\Delta_1) \times X(\Delta_2)$.

Suppose that $\ell_i : \widehat{M}_i \rightarrow \mathbb{Z}$ are lattice homomorphisms such that $(\ell_i)_{\mathbb{R}}(\Delta_i) \subset [0, 1]$. We pull ℓ_i to the product of lattices and on $\widehat{M}_1 \times \widehat{M}_2$ we define the form $(\ell_1 - \ell_2)$. The form defines a diagonal action $\lambda^{\ell_1 - \ell_2}$ of \mathbb{C}^* on $X^\times \subset \mathbb{P}^{n_1 n_2 - 1}$ which on the coordinate associated to $\chi^{(u_1, u_2)}$, where $u_i \in \Delta_i \cap \widehat{M}_i$, has the weight $\ell_1(u_1) - \ell_2(u_2) \in \{-1, 0, 1\}$. Accordingly, we regroup the coordinates of $\mathbb{P}^{n_1 n_2 - 1}$ and write them as $[z_i^-, z_j^0, z_k^+]$ depending on whether they are of weight $-1, 0$ and 1 , respectively. That is

$$\lambda_{\ell_1 - \ell_2}(t)[z_i^-, z_j^0, z_k^+] = [t^{-1}z_i^-, z_j^0, tz_k^+]$$

The above formula defines the action of $\lambda_{\ell_1 - \ell_2}$ on the cone over X^\times and thus a \mathbb{C}^* -linearization of the bundle $\mathcal{O}_{X^\times}(1)$ in the sense of GIT. By X^0 let us denote the intersection of X^\times with the complement of the space spanned on the eigenvectors of $\lambda_{\ell_1 - \ell_2}$ of weight $\neq 0$, that is $X^0 = X^\times \setminus \{[z_i^-, z_j^0, z_k^+] : \forall j \ z_j^0 = 0\}$

We set $\widehat{M} = \ker(\ell_1 - \ell_2)$ and $\Delta = \Delta^\times \cap \ker(\ell_1 - \ell_2) = \Delta_1 \times_{\ell_1 \times \ell_2} \Delta_2$. By A.4 Δ is a normal polytope and by $X(\Delta)$ we denote its associated toric variety.

Proposition 2.26. *In the above situation the set X^0 is equal to the set of the semistable points of the action of $\lambda^{\ell_1 - \ell_2}$. The projection to the weight 0 eigenspace $[z_i^0, z_j^0, z_k^+] \mapsto [z_j^0]$ defines a regular map of X^0 to $X(\Delta)$ and $X(\Delta)$ is a good quotient for the action of $\lambda_{\ell_1 - \ell_2}$.*

Proof. The sections of $\mathcal{O}_{X^\times}(m)$ for $m > 0$ make a vector space spanned on χ^u , where $u \in m\Delta^\times \cap M^\times$. Among them, these which are invariant with respect to the action of $\lambda^{\ell_1 - \ell_2}$ are associated to u 's in the intersection with $\ker \ell_1 - \ell_2$ thus in $m\Delta \cap M$. By

the normality of Δ , see A.4, the algebra of invariant sections is generated by these from $\mathcal{O}_{X^\times}(1)$. Thus the set of semistable points of the action of $\lambda_{\ell_1-\ell_2}$ is where at least one of the coordinates z_j^0 is non-zero and the quotient map is the projection to the weight zero eigenspace. \square

Corollary 2.27. *Let (\mathcal{T}_1, ℓ_1) and (\mathcal{T}_2, ℓ_2) be two pointed trees. Then $X(\mathcal{T}_1 \vee_{\ell_2} \mathcal{T}_2)$ is a good quotient of $X(\mathcal{T}_1) \times X(\mathcal{T}_2)$ with respect to an action of $\lambda^{\ell_1-\ell_2}$.*

Example 2.28. Consider the \mathbb{C}^* action on the product $\mathbb{P}_1^3 \times \mathbb{P}_2^3$ given by the formula:

$$\lambda(t)([z_0^1, z_1^1, z_2^1, z_3^1], [z_0^2, z_1^2, z_2^2, z_3^2]) = ([z_0^1, tz_1^1, tz_2^1, z_3^1], [z_0^2, t^{-1}z_1^2, t^{-1}z_2^2, z_3^2])$$

where the superscripts of the coordinates indicate the factor in the product $\mathbb{P}_1^3 \times \mathbb{P}_2^3$. The following rational map $\mathbb{P}_1^3 \times \mathbb{P}_2^3 \dashrightarrow \mathbb{P}^7$ is λ equivariant and regular outside the set $\{z_0^1 = z_3^1 = z_2^1 = z_2^2 = 0\} \cup \{z_1^1 = z_1^2 = z_0^2 = z_3^2 = 0\}$, each component of this set is a quadric $\mathbb{P}^1 \times \mathbb{P}^1$:

$$([z_0^1, z_1^1, z_2^1, z_3^1], [z_0^2, z_1^2, z_2^2, z_3^2]) \mapsto [z_0^1 z_0^2, z_0^1 z_3^2, z_1^1 z_1^2, z_1^1 z_2^2, z_2^1 z_1^2, z_2^1 z_2^2, z_3^1 z_0^2, z_3^1 z_3^2]$$

If $[x_0, \dots, x_7]$ are coordinates in \mathbb{P}^7 then the image of this map is the intersection of two quadrics $\{x_0 x_7 = x_1 x_6\} \cap \{x_2 x_5 = x_3 x_4\}$.

The above claim will be clear if we write functions $z_i^1 z_j^2$ in terms of characters of the respective torus, which we denote by e_i^1 and e_j^2 , respectively. Namely, dividing the right hand side of the above displayed formula by $z_0^1 z_0^2$ we get the following sequence of rational functions:

$$[1, \chi^{e_1^1+e_2^2}, \chi^{e_0^1+e_1^1} \chi^{e_0^2+e_2^2}, \chi^{e_0^1+e_1^1} \chi^{e_0^2+e_2^2}, \chi^{e_0^1+e_1^1} \chi^{e_0^2+e_2^2}, \chi^{e_0^1+e_1^1} \chi^{e_0^2+e_2^2}, \chi^{e_1^1+e_2^2}, \chi^{e_1^1+e_2^2} \chi^{e_1^1+e_2^2}]$$

If we write the sums of the exponents of the above rational functions in $M_1 \oplus M_2$ and call $e_0 = e_0^1 + e_0^2$ then we get the vertices of $\Delta(\mathcal{T}^{*3} \vee \mathcal{T}^{*3})$ which we computed in example 2.19. From the above formula we can read the weights with which 1-parameter groups $\lambda_{(e_j^i)^*}$, for $i, j = 1, 2$, associated to leaves, act on the quotient variety in \mathbb{P}^7 .

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3 3-valent binary trees.

Summary: From this point on we concentrate on understanding varieties associated to 3-valent binary trees and we prove main results of the present note which are as

follows: (1) such varieties have only Gorenstein terminal singularities and are Fano of index 4, (2) any two such varieties associated to trees with the same number of leaves are in the same connected component of the Hilbert scheme of the projective space, (3) their Hilbert-Ehrhard polynomial can be computed effectively..

3.1 Paths, networks and sockets.

Let \mathcal{T} be a 3-valent binary symmetric tree. In section 2.2 we identified the variety $X(\mathcal{T})$ in $\mathbb{P}^{2^{|\mathcal{L}|-1}-1}$ with the closure of the image of a torus map defined by a polytope $\Delta(\mathcal{T})$. We recall that the linear coordinates on the ambient projective space can be identified with the vertices of $\Delta(\mathcal{T})$ which are among the vertices of the cube \square_M satisfying parity relation with respect to the forms $v \in \mathcal{N} \subset N$, 2.9. For 3-valent trees we have a convenient interpretation of these points.

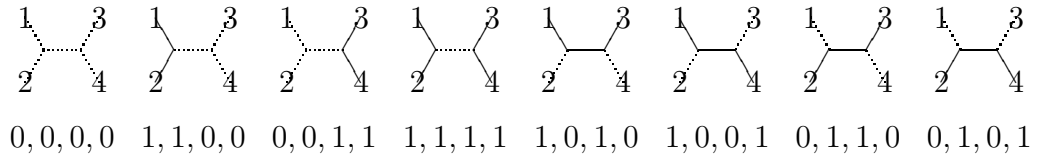
Definition 3.1. *Let \mathcal{T} be a 3-valent tree. A path γ on \mathcal{T} of length $m \geq 1$ is a choice of $m+1$ distinct vertices v_0, \dots, v_m such that v_0 and v_m are leaves (called the ending points of γ) and there exists m edges, e_1, \dots, e_m such that for $i = 1, \dots, m$ it holds $\partial(e_i) = \{v_{i-1}, v_i\}$.*

A network of paths (or just a network) Γ on \mathcal{T} is a set of paths (possibly an empty set), each two of them have no common vertex (neither edge). For any network of paths Γ on \mathcal{T} we define the socket $\mu(\Gamma) \subset \mathcal{L}$ to be the set of leaves which are ending points of paths in Γ .

A tree \mathcal{T} is labeled if its leaves are numbered by $1, \dots, |\mathcal{L}|$. A subset $\mu \subset \mathcal{L}$ is represented by a characteristic sequence $\kappa(1), \dots, \kappa(|\mathcal{L}|)$ in which $\kappa(i) = 1$ or 0, depending on whether the leaf numbered by i is in μ or not.

Sockets of networks will be identified by their characteristic binary sequences. We note that, clearly, every socket consists of even number of elements in \mathcal{L} .

Example 3.2. Let us consider a labeled 3-valent tree with four leaves. In the following diagram, in the upper row we draw all possible networks on this tree, where paths are denoted by solid line segments. In the lower row we write down the respective sockets in terms of characteristic sequences of length four



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Lemma 3.3. *Let \mathcal{T} be a 3-valent tree. Associating to a network Γ a point $u(\Gamma) = \sum_e \Gamma(e) \cdot e \in M(\mathcal{T})$, where $\Gamma(e) = 1, 0$ depending on whether e is on Γ or not, defines a bijection between networks and vertices of $\Delta(\mathcal{T})$*

Proof. First note that $u(\Gamma) \in \Delta(\mathcal{T})$. To define the inverse of $\Gamma \rightarrow u(\Gamma)$, for any vertex $u = \sum_{e \in \mathcal{E}} \epsilon_e \cdot e \in \Delta(\mathcal{T})$ we define the support of u consisting of edges of \mathcal{T} whose contribution to u is nonzero, i.e. $\{e \in \mathcal{E} : \epsilon_e(u) = 1\}$. The parity condition $\forall_{v \in \mathcal{N}}$ either $v(u) = 0$ or $v(u) = 2$ yields that these edges define a network on \mathcal{T} . \square

We note that, because of 2.11, there are $2^{|\mathcal{L}|-1}$ networks. On the other hand, the association of the socket to a network gives a map from the set of networks to the subsets of leaves. This map is surjective, that is, every subset μ of \mathcal{L} with even number of elements is a socket of a network. Indeed, this follows by a straightforward induction with respect to the number of leaves of the tree: in the induction step we write a tree \mathcal{T}_{n+1} with $n+1$ leaves as a graft of a tree \mathcal{T}_n with n leaves and a star tree with 3 leaves and consider three cases depending on how many of the two new leaves replacing one old are in the set $\mu \subset \mathcal{L}$.

Finally, because the number of all subsets of \mathcal{L} with even number of elements equals to $2^{|\mathcal{L}|-1}$ we get the following.

Lemma 3.4. *Let \mathcal{T} be a 3-valent tree. Then associating to a network its socket defines a bijection between the set of networks of paths on \mathcal{T} and the set of subsets of \mathcal{L} which have even number of elements.*

We note that the sockets of a tree \mathcal{T} form a convenient basis in the space $W_{\mathcal{L}}^{\rho}$, which was introduced in section 1.2. Indeed, in order to use toric arguments we have diagonalized the action of the involution ρ on W with a basis ν_0, ν_1 such that $\rho(\nu_i) = (-1)^i \nu_i$. Now any socket (or, equivalently, a subset of \mathcal{L} with even number of elements) whose characteristic binary function is $\kappa : \mathcal{L} \rightarrow \{0, 1\}$, defines an element $\otimes_{v \in \mathcal{L}} \nu_{\kappa(v)}^*$ in $(W_{\mathcal{L}}^{\rho})^*$. Similarly, to any network Γ on \mathcal{T} we associate a vector $\otimes_{e \in \mathcal{E}} \omega_{\Gamma(e)}^*$ in $\widehat{W}^{\mathcal{E}}$, where ω_i is such $\widehat{\rho}(\omega_i) = (-1)^i \omega_i$ and $\Gamma(e) = 1, 0$ depending on whether e is in Γ or not. Now associating to a network its sockets defines an isomorphism $(\widehat{W}^{\mathcal{E}})^{G_{\mathcal{N}}} \rightarrow (W_{\mathcal{L}}^{\rho})$ which one can compare to what we discuss in 1.12.

We have a convenient description of the action of one-parameter groups associated to leaves of \mathcal{T} in terms of socket coordinates of $\mathbb{P}(W_{\mathcal{L}}^{\rho})$. Namely, given a leaf ℓ the 1-parameter group λ^{ℓ} acts on the coordinate χ^{κ} with the weight $\kappa(\ell)$.

Construction 3.5. Using networks and sockets, and the toric formalism, one can explain the inclusion $X(\mathcal{T}) \subset \mathbb{P}(W_{\mathcal{L}}^{\rho})$ as follows. Let $\widetilde{M} = \bigoplus_{\kappa \neq 0} \mathbb{Z} \cdot \kappa$ be a lattice, a free abelian group generated by non-empty sockets of a tree \mathcal{T} . The empty socket $\kappa = 0$ we interpret as the zero of the lattice. Then $\mathbb{P}(W_{\mathcal{L}}^{\rho})$ is a toric variety $X(\widetilde{\Delta}^0)$ associated to a unit simplex $\widetilde{\Delta}^0$ in \widetilde{M} spanned on the vectors of the distinguished basis.

Now the bijective map *sockets* \leftrightarrow *networks* gives rise to a homomorphism of lattices $\widetilde{M} \rightarrow \widetilde{M}$, where, recall, the latter lattice is spanned in M by the points associated to networks. This gives a surjective map from the symmetric graded algebra spanned by all the sockets, which is just algebra of polynomials $\mathbb{C}[\chi^{\kappa}]$, to the algebra $A(\Delta)$, hence we get the inclusion $X(\mathcal{T}) \subset \mathbb{P}(W_{\mathcal{L}}^{\rho})$, c.f. 2.3 and 2.23.

Definition 3.6. Let Δ be a normal lattice polytope in a lattice M . Let us choose two collection of points u_1, \dots, u_r and w_1, \dots, w_s in $\Delta \cap M$ and positive integers a_1, \dots, a_r and b_1, \dots, b_s . This data defines a relation of degree d for Δ if $a_1 + \dots + a_r = b_1 + \dots + b_s = d$ and

$$a_1 u_1 + \dots + a_r u_r = b_1 w_1 + \dots + b_s w_s$$

The relation is called *primitive* if $\{u_1, \dots, u_r\} \cap \{w_1, \dots, w_s\} = \emptyset$.

Let us recall that given the projective variety $X \subset \mathbb{P}^r$ with graded coordinate ring $S(X) = \bigoplus_{m \geq 0} S^m(X)$ its ideal $\mathcal{I}(X)$ is the kernel of evaluation map $\text{Sym}(S^1(X)) \rightarrow S(X)$. The following result is known as binomial generation of a toric ideal, see [Eisenbud, Sturmfels], [Sturmfels].

Lemma 3.7. Suppose that we are in the situation of 2.23. Then the ideal $\mathcal{I}(X(\Delta))$ is generated by polynomials

$$(\chi^{u_1})^{a_1} \dots (\chi^{u_r})^{a_r} - (\chi^{w_1})^{b_1} \dots (\chi^{w_s})^{b_s}$$

where u_1, \dots, u_r and w_1, \dots, w_s , together with a_1, \dots, a_r and b_1, \dots, b_s define a primitive relation for Δ .

Example 3.8. The following are primitive relations and respective equations for the polytope coming from a 3-valent tree with four leaves, c.f. example 3.2. First, we describe them in terms of networks; they are as follows:

$$\begin{array}{c} 1 \\ \diagup \\ \text{---} \\ \diagdown \\ 2 \end{array} \begin{array}{c} 3 \\ \diagdown \\ \text{---} \\ \diagup \\ 4 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 2 \end{array} \begin{array}{c} 3 \\ \diagup \\ \text{---} \\ \diagdown \\ 4 \end{array} = \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 2 \end{array} \begin{array}{c} 3 \\ \diagup \\ \text{---} \\ \diagdown \\ 4 \end{array} + \begin{array}{c} 1 \\ \diagup \\ \text{---} \\ \diagdown \\ 2 \end{array} \begin{array}{c} 3 \\ \diagdown \\ \text{---} \\ \diagup \\ 4 \end{array}$$

and

$$\begin{array}{c} 1 \\ \diagup \\ 2 \end{array} \begin{array}{c} 3 \\ \diagdown \\ 4 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 3 \\ \diagup \\ 4 \end{array} = \begin{array}{c} 1 \\ \diagup \\ 2 \end{array} \begin{array}{c} 3 \\ \diagup \\ 4 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 3 \\ \diagdown \\ 4 \end{array}$$

On $\mathbb{P}(W_{\mathcal{L}}^{\rho})$ we introduce coordinates $x_{\kappa(1)\dots\kappa(4)}$ indexed by characteristic sequences for the sockets in \mathcal{L} . Then the respective equations defining $X(\mathcal{T})$ are as follows:

$$\begin{aligned} x_{0000} \cdot x_{1111} &= x_{1100} \cdot x_{0011} \\ x_{1001} \cdot x_{0110} &= x_{1010} \cdot x_{0101} \end{aligned}$$

Finally, let us note that renumbering the leaves or, equivalently, changing the shape of a 3-valent tree connecting the four numbered leaf vertices, produces the following respective equations

$$\begin{array}{cc} \begin{array}{c} 1 \\ \diagup \\ 3 \end{array} \begin{array}{c} 2 \\ \diagdown \\ 4 \end{array} & \begin{aligned} x_{0000} \cdot x_{1111} &= x_{1010} \cdot x_{0101} \\ x_{1001} \cdot x_{0110} &= x_{1100} \cdot x_{0011} \end{aligned} \\ \\ \begin{array}{c} 1 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 3 \end{array} & \begin{aligned} x_{0000} \cdot x_{1111} &= x_{1001} \cdot x_{0110} \\ x_{1001} \cdot x_{0110} &= x_{1010} \cdot x_{0101} \end{aligned} \end{array}$$

We note that all the above equations involve only four quadratic monomials: $x_{0000}x_{1111}$, $x_{1100}x_{0011}$, $x_{1010}x_{0101}$, $x_{1001}x_{0110}$. Moreover, given any leaf ℓ , the 1-parameter group λ^{ℓ} acts with weight 1 on each of them.

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3.2 Dual polytopes, fans, resolutions and Fano varieties.

In the situation of 2.22 the description of the fan of the variety $X(\Delta)$ in \widehat{N} is given in terms of its support functions [Oda, Thm. 2.22] or dual polytopes [Fulton].

Example 3.9. By looking at the example 2.16 and the inequalities which appear there we see that the fan of $X(\mathcal{T}^3)$ in $\widehat{N} \supset N$ has rays generated by the following elements: $-v/2 = -(e_0^* + e_1^* + e_2^*)/2$, $v/2 - e_0^* = (e_1^* + e_2^* - e_0^*)/2$, $v/2 - e_1^* = (e_0^* + e_2^* - e_1^*)/2$, $v/2 - e_2^* = (e_0^* + e_1^* - e_2^*)/2$.

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The formula from 2.20 can be used to get the description of the polytope dual to $\Delta(\mathcal{T})$, hence to describing the fan of $X(\mathcal{T})$ for 3-valent trees.

Lemma 3.10. *Let \mathcal{T} be a 3-valent binary symmetric tree with n inner nodes. Then the polytope $\Delta(\mathcal{T})$ is defined in $M_{\mathbb{R}}$ by $4n$ inequalities, which are as follows: for any inner node $v \in \mathcal{N}$, such that $v = e_{v,1}^* + e_{v,1}^* + e_{v,2}^*$ we take*

$$(-v/2)(\cdot) \geq -1, \quad (v/2 - e_{v,0}^*)(\cdot) \geq 0, \quad (v/2 - e_{v,1}^*)(\cdot) \geq 0, \quad (v/2 - e_{v,2}^*)(\cdot) \geq 0.$$

Proof. Let (\mathcal{T}_1, ℓ_1) and (\mathcal{T}_2, ℓ_2) be pointed trees. If $\Delta_i = \Delta(\mathcal{T}_i) \subset (M_i)_{\mathbb{R}}$ is defined by inequalities with respect to some forms w_j^i in $(N_i)_{\mathbb{R}}$ then $\Delta_1 \times \Delta_2$ is defined by forms $(w_j^1, 0)$ and $(0, w_j^2)$ in $(N_1)_{\mathbb{R}} \times (N_2)_{\mathbb{R}}$. Then the classes of these forms in $N = (N_1 \times N_2)_{\mathbb{R}} / \mathbb{R}(\ell_1 - \ell_2)$ define the fiber product of Δ_i 's. \square

Definition 3.11. For a binary symmetric 3-valent tree \mathcal{T} we define a polytope $\Delta^\vee(\mathcal{T})$ in $N_{\mathbb{R}}$ which is the convex hull of $-v/2 = -(e_{v,0}^* + e_{v,1}^* + e_{v,2}^*)/2$, $v/2 - e_{v,0}^* = (e_{v,1}^* + e_{v,2}^* - e_{v,0}^*)/2$, $v/2 - e_{v,1}^* = (e_{v,0}^* + e_{v,2}^* - e_{v,1}^*)/2$, $v/2 - e_{v,2}^* = (e_{v,0}^* + e_{v,1}^* - e_{v,2}^*)/2$, for $v \in \mathcal{N}$ and $e_{v,0}, e_{v,1}, e_{v,2}$ edges containing v .

Let us note that the listed above points are in fact vertices of $\Delta^\vee(\mathcal{T})$. Indeed, take $v \in \mathcal{N}$ and $e_{v,0}, e_{v,1}, e_{v,2}$ the edges containing v . Then by looking at the points which span $\Delta^\vee(\mathcal{T})$ we see that $(e_{v,0} + e_{v,1} + e_{v,2})(\Delta^\vee(\mathcal{T})) \geq -3/2$ with the equality only for the point $-(e_{v,0}^* + e_{v,1}^* + e_{v,2}^*)/2$ which therefore is a vertex. Similarly, $(e_{v,0} + e_{v,1} - e_{v,2})(\Delta^\vee(\mathcal{T})) \leq 3/2$ with the equality only for $(e_{v,0}^* + e_{v,1}^* - e_{v,2}^*)/2$.

Lemma 3.12. Let $\hat{\sigma} = \sum_{e \in \mathcal{E}} e$ then $4\Delta(\mathcal{T}) - 2\hat{\sigma}$ and $\Delta^\vee(\mathcal{T})$ are dual, or polar, one to another in the sense that

$$\begin{aligned} \Delta^\vee(\mathcal{T}) &= \{w \in N_{\mathbb{R}} : w(4\Delta(\mathcal{T}) - 2\hat{\sigma}) \geq -1\} \\ 4\Delta(\mathcal{T}) - 2\hat{\sigma} &= \{u \in M_{\mathbb{R}} : u(\Delta^\vee(\mathcal{T})) \geq -1\} \end{aligned}$$

Proof. The first equality is a restatement of 3.10, the second equality follows because the polar polytope of the polar is the original polytope, [Fulton, Sect 1.5]. \square

Notation 3.13. For a vertex of $\Delta(\mathcal{T})$ we define its dual face $u^\perp = \Delta^\vee(\mathcal{T}) \cap \{w : w(4u - 2\hat{\sigma}) = -1\}$. By \tilde{u}^\perp we will understand the polytope which is the convex hull of u^\perp and $0 \in N_{\mathbb{R}}$ while by \hat{u}^\perp we will understand the cone spanned in $N_{\mathbb{R}}$ by u^\perp .

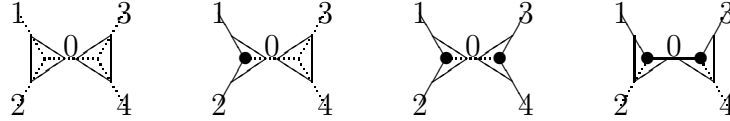
Let u be a vertex of $\Delta(\mathcal{T})$ which we can represent as a network of paths, $\Gamma(u)$. Then $v(u)$ is either 0 or 2, depending on whether $\Gamma(u)$ contains v and, similarly $e^*(u)$ is, respectively 0 or 1. Thus $(-v/2)(4u - 2\hat{\sigma}) = -1$ if v is in $\Gamma(u)$ and $(-v/2)(4u - 2\hat{\sigma}) = 3$ otherwise. On the other hand $(v/2 - e_{v,0}^*)(4u - 2\hat{\sigma}) = -1$ if either v is not in $\Gamma(u)$ or if both v and $e_{v,0}^*$ are in $\Gamma(u)$. Finally, $(v/2 - e_{v,0}^*)(4u - 2\hat{\sigma}) = 3$ if v is in $\Gamma(u)$ but $e_{v,0}^*$ is not.

Therefore, for any vertex u of $\Delta(\mathcal{T})$ and any node $v \in \mathcal{N}$ exactly three of the following four points $-v/2 = -(e_{v,0}^* + e_{v,1}^* + e_{v,2}^*)/2$, $v/2 - e_{v,0}^* = (e_{v,1}^* + e_{v,2}^* - e_{v,0}^*)/2$, $v/2 - e_{v,1}^* = (e_{v,0}^* + e_{v,2}^* - e_{v,1}^*)/2$, $v/2 - e_{v,2}^* = (e_{v,0}^* + e_{v,1}^* - e_{v,2}^*)/2$ are in u^\perp which therefore has $3n$ vertices.

Example 3.14. We will visualize the points of \widehat{N} on the graph of the tree in the following way. Given a 3-valent node v with edges $e_{v,0}$, $e_{v,1}$, $e_{v,2}$, which for simplicity we denote just by numbers on the graph, the point $-v/2$ will be denoted by the dot at the vertex, while the point $v/2 - e_{v,0}^*$ by the secant opposing the edge $e_{v,0}$, that is

$$\begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \\ | \\ 0 \end{array} \quad \text{and} \quad \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \\ | \\ 0 \end{array}, \quad \text{respectively}$$

Using this notation we can put on the same picture both, the system of paths associated to a vertex u of $\Delta(\mathcal{T})$ as well as the respective points in u^\perp . We put only four out of eight systems of paths from 3.2 since the other ones are obtained by renumbering of leaves.



In each of these cases the polytope \widetilde{u}^\perp can be divided into two simplexes, each of them having edges which make a basis of the lattice \widehat{N} . For example:

$$\begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \\ | \\ 0 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \\ | \\ 0 \end{array} = \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \\ | \\ 0 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \\ | \\ 0 \end{array} \cup \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \\ | \\ 0 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \\ | \\ 0 \end{array} = \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \\ | \\ 0 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \\ | \\ 0 \end{array} \cup \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \\ | \\ 0 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \\ | \\ 0 \end{array}$$

The first equality means that \widetilde{u}^\perp in this case is a union of a simplex with edges $(e_1^* + e_2^* - e_0^*)/2$, $(e_0^* + e_2^* - e_1^*)/2$, $(e_0^* + e_1^* - e_2^*)/2$, $(e_3^* + e_4^* - e_0^*)/2$, $(e_0^* + e_3^* - e_4^*)/2$ and another one with edges $(e_1^* + e_2^* - e_0^*)/2$, $(e_0^* + e_2^* - e_1^*)/2$, $(e_0^* + e_1^* - e_2^*)/2$, $(e_3^* + e_4^* - e_0^*)/2$, $(e_0^* + e_4^* - e_3^*)/2$. The common part of these two simplexes is a simplex with edges $(e_1^* + e_2^* - e_0^*)/2$, $(e_0^* + e_2^* - e_1^*)/2$, $(e_0^* + e_1^* - e_2^*)/2$, $(e_3^* + e_4^* - e_0^*)/2$, which contains $e_0^*/2 = ((e_0^* + e_3^* - e_4^*)/2 + (e_0^* + e_4^* - e_3^*)/2)/2$.

This example is even more transparent when we write \widehat{N} as a sum of a rank 2 lattice spanned by $(e_1^* + e_2^* - e_0^*)/2$ and $(e_3^* + e_4^* - e_0^*)/2$, and of rank 3 lattice spanned by $(e_0^* + e_2^* - e_1^*)/2$, $(e_0^* + e_1^* - e_2^*)/2$, $(e_0^* + e_3^* - e_4^*)/2$ which contains also $(e_0^* + e_4^* - e_3^*)/2$. Then our division of the cone \widehat{u}^\perp comes by multiplying by the cone $\mathbb{R}_{\geq 0}(e_1^* + e_2^* - e_0^*) + \mathbb{R}_{\geq 0}(e_3^* + e_4^* - e_0^*)$ the standard division of the 3-dimensional cone generated by $(e_0^* + e_2^* - e_1^*)/2$, $(e_0^* + e_1^* - e_2^*)/2$, $(e_0^* + e_3^* - e_4^*)/2$ and $(e_0^* + e_4^* - e_3^*)/2$, see [Fulton, p. 49], which in geometric terms is a small resolution of a 3-dimensional quadric cone singularity giving rise to so-called Atiyah flop.

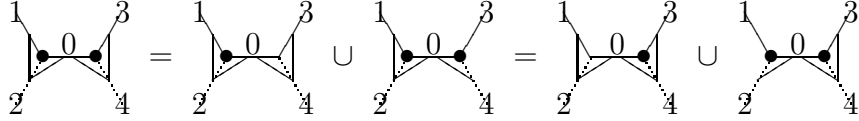
The same argument works whenever $\Gamma(u)$ does not contain e_0 . Then u^\perp contains $(e_0^* + e_2^* - e_1^*)/2$, $(e_0^* + e_1^* - e_2^*)/2$, $(e_0^* + e_4^* - e_3^*)/2$, $(e_0^* + e_3^* - e_4^*)/2$ and we can make a similar division of \widetilde{u}^\perp using the equality

$$(e_0^* + e_2^* - e_1^*)/2 + (e_0^* + e_1^* - e_2^*)/2 = (e_0^* + e_4^* - e_3^*)/2 + (e_0^* + e_3^* - e_4^*)/2$$

If $\Gamma(u)$ contains e_0 then we use the identity

$$-(e_0^* + e_1^* + e_2^*)/2 + (e_1^* + e_2^* - e_0^*)/2 = -(e_0^* + e_3^* + e_4^*)/2 + (e_3^* + e_4^* - e_0^*)/2$$

which presents $-e_0^*/2 \in u^\perp$ as an average of two different pairs of vertices to make a similar decomposition



◦

Now we shall show that the above discussion can be generalized to the case of trees with more inner nodes.

Lemma 3.15. *Suppose that \mathcal{T} is a binary symmetric 3-valent tree with n inner nodes. For any u , a vertex of $\Delta(\mathcal{T})$ there exists a division of u^\perp (or, equivalently of \tilde{u}^\perp) into a union of 2^{n-1} (normalized) volume 1 simplexes. Equivalently, the cone \tilde{u}^\perp can be divided into a union of simplicial cones which are regular (i.e. their generators form bases of \hat{N}).*

Proof. The construction of the division will proceed along an ascending sequence of subtrees of \mathcal{T} , starting from an inner node of \mathcal{T} . That is we have an ascending sequence of 3-valent trees

$$\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots \subset \mathcal{T}_{n-1} \subset \mathcal{T}_n = \mathcal{T}$$

where \mathcal{T}_i has i inner nodes and \mathcal{T}_{i+1} is obtained from \mathcal{T}_i as a graft with a star 3-valent tree. Forgetting of edges which are not in \mathcal{T}_i gives a sequence of surjective maps $M(\mathcal{T}) \rightarrow \cdots \rightarrow M(\mathcal{T}_i) \rightarrow \cdots \rightarrow M(\mathcal{T}_1)$ which implies a sequences of inclusions $\hat{N}(\mathcal{T}_1) \subset \cdots \subset \hat{N}(\mathcal{T}_i) \subset \cdots \subset \hat{N}(\mathcal{T}_n)$. The restriction of the networks of paths u to \mathcal{T}_i is a network on \mathcal{T}_i as well we will denote it by u_i . Clearly $u^\perp \cap N(\mathcal{T}_i)_\mathbb{R} = u_i^\perp$.

Now we will define the division of u_i^\perp inductively. The polytope u_i^\perp is just a simplex so let us assume that $u_i^\perp = \sum \delta_i^j$ where $j = 1, \dots, 2^{i-1}$ and the normalized volume of δ_i^j with respect to the lattice $\hat{N}(\mathcal{T}_i)$ is 1. Let v^i be an inner node of \mathcal{T}_{i+1} which was a leaf of \mathcal{T}_i , let e_0^i be a petiole of \mathcal{T}_i which become an inner edge of \mathcal{T}_{i+1} and let e_1^i and e_2^i are the two new petioles of \mathcal{T}_{i+1} which contain v^i .

Now we make argument as in 3.14. If e_0^i is in u then $-(e_0^i)^*/2 \in u_i^\perp$ and we may assume that e_1^i is in u and e_2^i is not. Now from any simplex δ_i^j from the original division of u_i^\perp we produce two simplexes by adding a new vertex at $((e_0^i)^* + (e_2^i)^* -$

$(e_1^i)^*/2$ and another one at either $-((e_0^i)^* + (e_1^i)^* + (e_2^i)^*)/2$ or at $((e_1^i)^* + (e_2^i)^* - (e_0^i)^*)/2$. Because

$$-((e_0^i)^* + (e_1^i)^* + (e_2^i)^*)/2 + ((e_1^i)^* + (e_2^i)^* - (e_0^i)^*)/2 = -(e_0^i)^*$$

and $-(e_0^i)^*/2 \in u_i^\perp$ this defines a good division of u_{i+1}^\perp .

If e_0^i is not in u then $(e_0^i)^*/2 \in u_i^\perp$ and we make a similar construction but now we have to consider two cases: either none of e_1^i, e_2^i is in u or both are in u . At either case the discussion is similar to that we encountered in 3.14. \square

In terms of toric geometry the division process implies the following.

Corollary 3.16. *The affine toric variety associated to the cone \widehat{u}^\perp has Gorenstein terminal singularities which admit a small resolution.*

Proof. The toric singularities are Cohen-Macaulay and since all the generators of the rays of \widehat{u}^\perp lie on the hyperplane $(4u - 2\sigma)(\cdot) = -1$ the singularities in question are Gorenstein. The division into regular simplicial cones involves adding no extra ray so the respective resolution is small which also implies that the original singularity is terminal. \square

We note that the construction of the division certainly depends on the choice of the root of the tree and changing the root gives a flop.

Let Σ be a fan in $\widehat{N}_\mathbb{R}$ consisting of cones \widehat{u}^\perp , where u is a vertex of $\Delta(\mathcal{T})$, and their faces. In other words, Σ contains cones spanned by the proper faces of $\Delta^\vee(\mathcal{T})$ (including the empty face, whose cone is the zero cone). Let us recall that equivariant line bundles on toric varieties are in a standard way described by piecewise linear functions on its fan, see [Oda, Sect. 2.1]. Setting $\Lambda|_{\widehat{u}^\perp} = -u$ we define a continuous piecewise linear function Λ on the fan Σ in $N_\mathbb{R}$ such that for every $v \in \mathcal{N}$ and $e_v \in \mathcal{E}$ containing v we have $\Lambda(-v/2) = -1$ and $\Lambda(v/2 - e_v^*) = 0$. The sections of the bundle related to Λ , see [Oda, Prop. 2.1], are in $\widehat{M} \cap \Delta(\mathcal{T})$. Therefore the toric variety $X(\Sigma)$ given by the fan Σ can be identified with the original variety $X(\Delta(\mathcal{T}))$ and the line bundle associated to Λ is $\mathcal{O}_X(1)$. On the other hand the function $4\Lambda - 2\sigma$ assumes value 1 on the primitive vectors in rays of Σ which allows us to identify the canonical divisor of $X(\Delta)$, see [Oda, Sect 2.1]. The result is the following.

Theorem 3.17. *Let \mathcal{T} be a 3-valent binary symmetric tree. Then the variety $X(\mathcal{T})$ is Gorenstein and Fano with terminal singularities. Moreover it is of index 4, that is the canonical divisor $K_{X(\mathcal{T})}$ is linearly equivalent to $\mathcal{O}_{X(\mathcal{T})}(-4)$.*

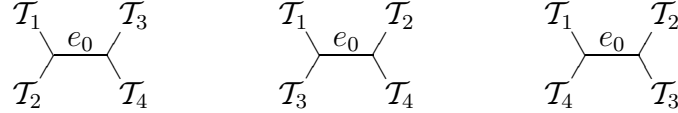
We note the following consequence of Kodaira-Kawamata-Viehweg vanishing, see e.g. [Kollár, Mori, Sect.2.5]

Corollary 3.18. *In the above situation $H^i(X(\mathcal{T}), \mathcal{O}(d)) = 0$ for $i > 0$ and $d \geq -3$. In particular for $d \geq 0$ we have $\dim_{\mathbb{C}} H^0(X(\mathcal{T}), \mathcal{O}(d)) = h_{X(\mathcal{T})}(d)$ where the latter is Poincare-Hilbert polynomial of $(X(\mathcal{T}), \mathcal{O}(1))$.*

3.3 Mutation of a tree, deformation of a model.

In example 3.8 we noted that a four-leaf 3-valent tree can be labeled in three non-equivalent ways. We can revert it to say that given four numbered leaves we have three 3-valent labeled trees connecting these leaves. By grouping in pairs the leaves whose petioles are attached to common inner nodes we can list these as follows: $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, and $(1, 4)(2, 3)$.

Now, given four pointed trees \mathcal{T}_i , where $i = 1, \dots, 4$ we can produce a tree \mathcal{T} by grafting the tree \mathcal{T}_i along the i -th leaf of a labeled 3-valent 4-leaf tree \mathcal{T}_0 . Here are possible configurations, e_0 denotes the inner edge of the tree \mathcal{T}_0



Definition 3.19. *In the above situation we say that there exists an elementary mutation along e_0 from one of the above trees to the other two. (We note that a mutation may actually yield an equivalent tree.) We say that two trees are mutation equivalent if there exists a sequence of elementary mutations from one to the other.*

Lemma 3.20. *Any two 3-valent trees with the same number of leaves are mutation equivalent.*

Proof. We prove, by induction, that any 3-valent tree is mutation equivalent to a caterpillar. To get the induction step it is enough to note that the graft of a caterpillar tree pointed at one of its legs with a star 3 valent tree contains a distinguished inner edge the mutation of which gives a caterpillar. \square

Now, let us recall the basics regarding deforming subvarieties in the projective space. Let \mathcal{B} be an irreducible variety (possibly non-complete). Consider the product $\mathbb{P}^m \times \mathcal{B}$ with the respective projections $p_{\mathbb{P}}$ and $p_{\mathcal{B}}$. Suppose that $\mathcal{X} \subset \mathbb{P}^m \times \mathcal{B}$ is a subscheme such that the induced projection $p_{\mathcal{B}|\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{B}$ is proper and flat. Suppose that for two points $a, b \in \mathcal{B}$ the respective scheme-theoretic fibers $X_a = \mathcal{X}_a$ and $X_b = \mathcal{X}_b$ are reduced and irreducible. Then we say that the subvariety X_a in \mathbb{P}^m

can be deformed to X_b over the base \mathcal{B} . This gives rise to a notion of deformation equivalent subvarieties of \mathbb{P}^m .

Definition 3.21. *Given two subvarieties X_1, X_2 in \mathbb{P}^m we say that they are deformation equivalent if their classes are in the same connected component of the Hilbert scheme of \mathbb{P}^m .*

Complete intersections of the same type are deformation equivalent. Let us consider an fundamental example, understanding of which is essential for the proof of the main result of this section.

Example 3.22. Let us consider \mathbb{P}^7 with homogeneous coordinates indexed by sockets of a 4-leaf tree \mathcal{T}_0 , as in example 3.8. In \mathbb{P}^7 we consider a family of intersections of 2 quadrics parametrized by an open subset \mathcal{B} of \mathbb{P}^2 with coordinates $[t_{(12)(34)}, t_{(13)(24)}, t_{(14)(23)}]$. We set $\mathcal{B} = \mathbb{P}^2 \setminus \{[1, \varepsilon, \varepsilon^2] : \varepsilon^3 = 1\}$ and over \mathcal{B} we consider \mathcal{X}^0 given in $\mathcal{B} \times \mathbb{P}^7$ by equations

$$\begin{aligned} t_{(12)(34)} \cdot x_{1100}x_{0011} + t_{(13)(24)} \cdot x_{1010}x_{0101} \\ + t_{(14)(23)} \cdot x_{1001}x_{0110} &= (t_{(12)(34)} + t_{(13)(24)} + t_{(14)(23)}) x_{0000}x_{1111} \\ \left(t_{(13)(24)} - t_{(14)(23)} \right) \cdot x_{1100}x_{0011} \\ + \left(t_{(14)(23)} - t_{(12)(34)} \right) \cdot x_{1010}x_{0101} \\ + \left(t_{(12)(34)} - t_{(13)(24)} \right) \cdot x_{1001}x_{0110} &= 0 \end{aligned}$$

Three special fibers of the projection $\mathcal{X}^0 \rightarrow \mathcal{B}$, namely $\mathcal{X}_{[1,0,0]}^0$, $\mathcal{X}_{[0,1,0]}^0$ and $\mathcal{X}_{[0,0,1]}^0$, are varieties associated to three 4-leaf trees labeled by (12)(34), (13)(24) and (14)(23), respectively. On the other hand \mathcal{X}^0 is a complete intersection of two quadrics and the map $\mathcal{X}^0 \rightarrow \mathcal{B}$ is equidimensional. The latter statement follows because over \mathcal{B} the matrix

$$\begin{bmatrix} t_{(12)(34)} & t_{(13)(24)} & t_{(14)(23)} & t_{(12)(34)} + t_{(13)(24)} + t_{(14)(23)} \\ t_{(13)(24)} - t_{(14)(23)} & t_{(14)(23)} - t_{(12)(34)} & t_{(12)(34)} - t_{(13)(24)} & 0 \end{bmatrix}$$

is of rank 2 hence any fiber over \mathcal{B} is a complete intersection of two non-proportional quadrics. Hence $\mathcal{X}^0 \rightarrow \mathcal{B}$ is flat because of [Eisenbud, Thm. 18.16].

By $T_0 \subset T_N$ denote the 4-dimensional subtorus associated to the lattice spanned by leaves, that is a subtorus of T_N with coordinates $\chi^{v_i^*}$, where $v_i, i = 1, \dots, 4$ are leaves of \mathcal{T}_0 . Torus T_0 acts on $\mathbb{P}^7 \times \mathcal{B}$ via the first coordinate, that is, for a leaf v_i of \mathcal{T}_0 and a socket κ we have $\lambda_{v_i}(t)(x_\kappa, t_{(\cdot)(\cdot)}) = t^{\kappa(v_i)}x_\kappa, t_{(\cdot)(\cdot)}$. Then by looking at the equations defining \mathcal{X}^0 we see that the inclusion $\mathcal{X}^0 \hookrightarrow \mathbb{P}^7 \times \mathcal{B}$ is equivariant with respect to this action.

We also note that a rational map $\mathbb{P}^7 - \rightarrow \mathbb{P}^3$, regular outside 16 linear \mathbb{P}^3 's, which is given by four quadrics:

$$[x_\kappa] \rightarrow [x_{0000}x_{1111}, x_{0011}x_{1100}, x_{0101}x_{1010}, x_{0110}x_{1001}]$$

defines a good quotient with respect to the action of T_0 on \mathbb{P}^7 , c.f.[Białyński-Birula, 7.1.1]. If we take a subvariety \mathcal{Z}^0 in the product $\mathbb{P}^3 \times \mathcal{B}$ defined by the equations

$$\begin{aligned} t_{(12)(34)} \cdot z_1 + t_{(13)(24)} \cdot z_2 + t_{(14)(23)} \cdot z_3 &= (t_{(12)(23)} + t_{(13)(24)} + t_{(14)(23)}) \cdot z_0 \\ (t_{(23)(14)} - t_{(14)(23)}) \cdot z_1 + (t_{(14)(23)} - t_{(12)(34)}) \cdot z_2 + (t_{(12)(34)} - t_{(23)(14)}) \cdot z_3 &= 0 \end{aligned}$$

then $\mathcal{Z}^0 \rightarrow \mathcal{B}$ is equidimensional and \mathcal{X}^0 is the fiber product of $\mathbb{P}^7 - \rightarrow \mathbb{P}^3$ and $\mathcal{Z}_0 \rightarrow \mathbb{P}^3$. As the result the induced rational map $\mathcal{X}^0 - \rightarrow \mathcal{Z}^0$ defines a good quotient of \mathcal{X}^0 with respect to the action of T_0 , [Białyński-Birula, 7.1.4].

◦

In what follows we construct an ambient variety which contains as locally complete intersections a flat family of varieties containing a geometric model of tree as well as models of the tree's elementary mutations.

Construction 3.23. Let \mathcal{T} be a tree with an inner edge e_0 which contains two 3-valent inner vertices. We can write \mathcal{T} as a graft of five trees: a labeled tree \mathcal{T}_0 with four leaves v_i , $i = 1, \dots, 4$, containing e_0 as an inner edge and four pointed trees (\mathcal{T}_i, ℓ_i) , with $i = 1, \dots, 4$ which are attached to \mathcal{T}_0 along the respectively labeled leaves. The edges in \mathcal{T} which have common nodes with e_0 we denote, respectively, by e_i , each e_i comes from a petiole of ℓ_i (or v_i). Recall, see 2.20, that $M(\mathcal{T})$ and $\Delta(\mathcal{T})$ can be expressed as fiber product of $M(\mathcal{T}_i)$ and $\Delta(\mathcal{T}_i)$, respectively. That is,

$$M(\mathcal{T}) = \prod_{i=0}^4 M(\mathcal{T}_i) \cap \bigcap_{i=1}^4 \ker(\ell_i - v_i), \quad \Delta(\mathcal{T}) = \prod_{i=0}^4 \Delta(\mathcal{T}_i) \cap \bigcap_{i=1}^4 \ker(\ell_i - v_i)$$

Now, as in 3.5, we consider the lattice \widetilde{M}_0 spanned on the non-trivial sockets of the tree \mathcal{T}_0 together with the unit simplex $\widetilde{\Delta}_0 \subset \widetilde{M}_0 \otimes \mathbb{R}$ and the maps $\widetilde{M}_0 \rightarrow M_0$ and $\widetilde{\Delta}_0 \rightarrow \Delta_0$ which give the inclusion $X(\mathcal{T}_0) \subset \mathbb{P}^7$ as a complete intersection of two quadrics. Forms v_i , $i = 1, \dots, 4$ pull-back to \widetilde{M}_0 and we denote them by \tilde{v}_i , respectively. Now we define

$$\overline{M} = \widetilde{M}_0 \times \prod_{i=1}^4 M(\mathcal{T}_i) \cap \bigcap_{i=1}^4 \ker(\ell_i - \tilde{v}_i) \quad \text{and}$$

$$\overline{\Delta} = \widetilde{\Delta}_0 \times \prod_{i=1}^4 \Delta(\mathcal{T}_i) \cap \bigcap_{i=1}^4 \ker(\ell_i - \tilde{v}_i)$$

As in 2.3 we define the toric variety $\mathcal{Y} = X(\overline{\Delta})$. We note that, by A.4 the polytope $\overline{\Delta}$ is normal in the lattice $\widetilde{M}_0 \times \prod_{i=1}^4 \widetilde{M}(\mathcal{T}_i) \cap \bigcap_{i=1}^4 \ker(\ell_i - \tilde{v}_i)$, which is spanned by its vertices. Also, by the construction we have the embeddings $X(\mathcal{T}) \hookrightarrow \mathcal{Y} \hookrightarrow \mathbb{P}(W_{\mathcal{L}}^\rho)$.

Lemma 3.24. *The inclusions*

$$\overline{M} \hookrightarrow \widetilde{M}_0 \times \prod_{i=1}^4 M(\mathcal{T}_i) \quad \text{and} \quad \overline{\Delta} \hookrightarrow \widetilde{\Delta}_0 \times \prod_{i=1}^4 \Delta(\mathcal{T}_i)$$

induce a rational map

$$\mathbb{P}^7 \times \prod_{i=1}^4 X(\mathcal{T}_i) \dashrightarrow \mathcal{Y}$$

which is a good quotient map (of the set over which it is defined) with respect to the action of the 4-dimensional torus T_0 generated by 1-parameter groups $\lambda_{v_i - \ell_i}$, where $i = 1, \dots, 4$. The subvariety

$$\widehat{\mathcal{X}} = \mathcal{X}^0 \times \prod_{i=1}^4 X(\mathcal{T}_i) \hookrightarrow \mathcal{B} \times \mathbb{P}^7 \times \prod_{i=1}^4 X(\mathcal{T}_i)$$

is T_0 equivariant and its quotient \mathcal{X} is locally complete intersection in $\mathcal{B} \times \mathcal{Y}$.

Proof. The first (quotient) part is the same as what we claim in 2.26, this time however we repeat the argument for all four fiber products in question.. The invariance of the variety $\widehat{\mathcal{X}}$ follows by the invariance of $\mathcal{X}^0 \hookrightarrow \mathcal{B} \times \mathbb{P}^7$ which we discussed in 3.22. Finally, since $\widehat{\mathcal{X}}$ is a complete intersection in $\mathcal{B} \times \mathbb{P}^7 \times \prod_{i=1}^4 X(\mathcal{T}_i)$ its image \mathcal{X} is a locally complete intersection in the quotient which is $\mathcal{B} \times \mathcal{Y}$, this follows from the definition of good quotient which locally is an affine quotient, [Białynicki-Birula, Ch. 5], hence functions defining $\widehat{\mathcal{X}}$ locally descend to functions defining \mathcal{X} . \square

Lemma 3.25. *Over an open set $\mathcal{B}' \subset \mathbb{P}^2$ containing points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$ the projection morphism $\mathcal{X} \rightarrow \mathcal{B}'$ is flat. The fibers over points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$ are reduced and isomorphic to, respectively, the geometric model of \mathcal{T} and of its elementary mutations along the edge e_0 .*

Proof. First we note that the fibers in question, $\mathcal{X}_{[* , * , *]}$, of $\mathcal{X} \rightarrow \mathcal{B}$ are geometric models as we claim. Indeed this follows from the universal properties of good quotients, c.f. [Białynicki-Birula], as they are quotients of the respective products $\mathcal{X}_{[* , * , *]}^0 \times \prod_{i=1}^4 X(\mathcal{T}_i)$, which are located, as three invariant subvarieties, in $\widehat{\mathcal{X}} = \mathcal{X}^0 \times \prod_{i=1}^4 X(\mathcal{T}_i)$. This, in particular, implies that the respective fibers of $\mathcal{X} \rightarrow \mathcal{B}$ are of the expected dimension, hence they are contained in a set $\mathcal{B}' \subset \mathbb{P}^2$ over which the map in question is equidimensional. Since \mathcal{Y} is toric it is Cohen-Macaulay and because \mathcal{X} is locally complete intersection in \mathcal{Y} , it is Cohen-Macaulay too [Eisenbud, Prop. 18.13]. Finally, the map $\mathcal{X} \rightarrow \mathcal{B}'$ is equidimensional hence it is flat, because \mathcal{B}' is smooth, see [Eisenbud, Thm. 18.16] \square

Theorem 3.26. *Geometric models of 3-valent trees with the same number of leaves are deformation equivalent in $\mathbb{P}(W_{\mathcal{L}}^p)$.*

Proof. This is a combination of 3.20 and of 3.25. □

3.4 Hilbert-Ehrhard polynomial.

Definition 3.27. *Given two pointed trees (\mathcal{T}_1, ℓ_1) and (\mathcal{T}_2, ℓ_2) we define a pointed graft which is a pointed tree $(\mathcal{T}, o) = (\mathcal{T}_1, \ell_1) \star (\mathcal{T}_2, \ell_2)$ where $\mathcal{T} = \mathcal{T}_1 \underset{\ell_1}{\vee}_{o_1} \mathcal{T}^{*3} \underset{o_2}{\vee}_{\ell_2} \mathcal{T}_2$, and o, o_1 and o_2 are the leaves of \mathcal{T}^{*3} .*

Example 3.28. Pointed graft of two 3-valent stars

$$\begin{array}{c} \diagup \circ \diagdown \end{array} \star \begin{array}{c} \circ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ \circ \end{array}$$

○

By arguments used in the proof of 2.20 we also get.

Proposition 3.29. *Let (\mathcal{T}_1, ℓ_1) and (\mathcal{T}_2, ℓ_2) be two pointed trees. Then*

$$\Delta(\mathcal{T}_1 \underset{\ell_1}{\vee}_{\ell_2} \mathcal{T}_2) = \Delta(\mathcal{T}_1) \underset{\ell_1 \times o_1}{\times} \Delta(\mathcal{T}^{*3}) \underset{o_2 \times \ell_2}{\times} \Delta(\mathcal{T}_2)$$

Let us consider a 3-dimensional lattice $M = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with a fixed tetrahedron Δ^0 with vertices $0, e_0 + e_1, e_0 + e_2, e_1 + e_2$. By $\widehat{M} \subset M$ we denote the index 2 sublattice spanned on the vertices of Δ^0 .

Definition 3.30. *Let n be a positive integer and let $f_1^n = f_1, f_2^n = f_2$ be two functions defined on the set $\{0, \dots, n\}$ with values in \mathbb{Z} or, more generally, in an arbitrary ring or algebra (we use the superscript n to indicate the domain of f 's). For any $k \in \{0, \dots, n\}$ we define*

$$(f_1 \star f_2)(k) = \sum_{\substack{u \in \widehat{M} \cap n\Delta^0 \\ e_0^*(u) = k}} (f_1(e_1^*(u)) \cdot f_2(e_2^*(u)))$$

We note that \star is commutative, that is $f_1^n \star f_2^n = f_2^n \star f_1^n$, but possibly not associative. By $(f^n)^{\star m}$ we denote the \star product of m copies of a chain of f^n , that is $f^n \star (f^n \star (\dots (f^n \star f^n) \dots))$. By 1^n we denote the constant function $\{0, \dots, n\} \rightarrow \{1\} \subset \mathbb{Z}$.

A function $f^n : \{0, \dots, n\} \rightarrow \mathbb{Z}$ will be called *symmetric* if $f^n(k) = f^n(n - k)$.

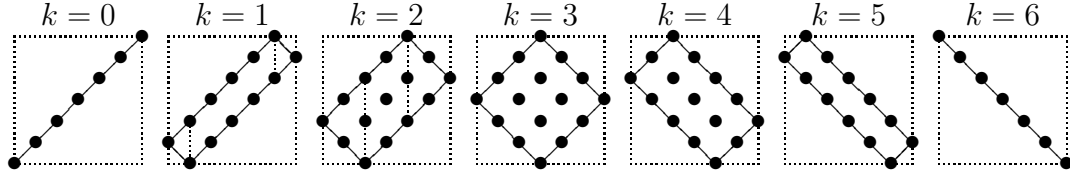
Lemma 3.31. *If $f_1 = f_1^n$, $f_2 = f_2^n : \{0, \dots, n\} \rightarrow \mathbb{Z}$ are symmetric functions then $f_1 \star f_2$ is a symmetric function as well and moreover for $k \leq n/2$ we have*

$$(f_1 \star f_2)(k) = 2 \cdot \left(\sum_{i=0}^{k-1} \sum_{j=0}^i f_1(i) f_2(k+i-2j) \right) + \left(\sum_{i=k}^{n-k} \sum_{j=0}^k f_1(i) f_2(k+i-2j) \right)$$

In particular, for $k \leq n/2$

$$(f_1 \star 1)(k) = 2 \sum_{i=0}^{k-1} (i+1) f_1(i) + \sum_{i=k}^{n-k} (k+1) f_1(i)$$

Proof. Let us look at the sections of the tetrahedron $n\Delta^0$ with hyperplanes $(e_0^*)^{-1}(k)$. We picture the situation for $n = 6$ and $k = 0, \dots, 6$, the dotted square is the section of the cube with the lower left corner satisfying relation $e_1^* = e_2^* = 0$, the section of the tetrahedron denoted with solid line and points inside the (closed) tetrahedron denoted by \bullet .



The definition of $f_1^n \star f_2^n$ is sum of the product of f_i^n 's over the lattice points of such a section. The sections over k and $n - k$ are obtained by a reflection with respect to either $e_1^* = 1/2$ or $e_2^* = 1/2$. Thus if one of f_i^n 's is symmetric then the $f_1^n \star f_2^n$ is symmetric as well.

On the other hand for $0 \leq k \leq n - k$ the tetrahedron's section is a rectangle with vertices $(k, 0)$, $(0, k)$, $(n - k, n)$, $(n, n - k)$ which we divide into two triangles and a parallelogram, the division is indicated by dotted vertical line segments for boxes labeled by $k = 1, 2$ in the above diagram. Because functions f_i^n are symmetric the values of the product $f_1^n \cdot f_2^n$ are the same for the points which are central symmetric with respect to the center of the square. Thus in the formula of the lemma we take the value $f_1(a)f_2(b)$ for all integral pairs (a, b) in the left hand side triangle and multiply it by 2 (that is the first summand in the formula) and add the sum over the parallelogram. \square

Example 3.32. We note that $(1^n)^{\star 2}(k) = (k+1)(n-k+1)$ is the number of lattice points in the rectangle used in the argument in the above proof of 3.31. On the

other hand by using the formula from 3.31 one gets

$$(1^n)^{\star 3}(k) = \frac{1}{6}(k+1)(n-k+1)(n^2 + kn - k^2 + 5n + 6)$$

◦

Let us recall that given a lattice polytope $\Delta \subset M_{\mathbb{R}}$ for any positive integer n we define Ehrhard function h_{Δ} as follows:

$$h_{\Delta}(n) = |(n \cdot \Delta \cap M)|$$

If Δ satisfies the assumptions of 2.3 then $h_{\Delta} = h_{X(\Delta)}$ where the latter is the Poincare-Hilbert polynomial of $(X(\Delta), \mathcal{O}(1))$ which, by definition, is equal to $\dim_{\mathbb{C}} H^0(X(\Delta), \mathcal{O}(m))$ for $m \gg 0$.

Definition 3.33. Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polytope which is not contained in any hyperplane and let $v \in N$ be a non-zero form on M . Suppose that $v(\Delta) \subset [0, 1]$. We define its relative Ehrhard function $f_{\Delta, v}^n : \{0, \dots, n\} \rightarrow \mathbb{Z}$ by setting

$$f_{\Delta, v}^n(k) = |v^{-1}(k) \cap n \cdot \Delta \cap M|$$

We note that, clearly, $\sum_{k=0}^n f_{\Delta, v}^n(k) = h_{\Delta}(n)$ is the usual Ehrhard function. Thus, in case of 2.3 the above definition can be restated in purely geometric fashion.

Lemma 3.34. Suppose that Δ satisfies assumptions of 2.3 and v is as in 3.33. Let us consider a linearization of the action of the 1-parameter group λ_v on $H^0(X(\Delta), \mathcal{O}(n))$ which has non-negative weights and the eigenspace of the zero weight is nontrivial. Then $f_{\Delta, v}^n(k)$ is equal to the dimension of the eigenspace of the action of λ_v of weight k .

Proof. This is a consequence of the standard properties of $X(\Delta)$, 2.23.4. □

Lemma 3.35. Let (\mathcal{T}_1, ℓ_1) and (\mathcal{T}_2, ℓ_2) be two pointed trees and let $f_{\ell_1}^n$ and $f_{\ell_2}^n$ be two relative Ehrhard functions associated to $\Delta(\mathcal{T}_1)$ and $\Delta(\mathcal{T}_2)$, respectively. If $(\mathcal{T}, o) = (\mathcal{T}_1, \ell_1) \star (\mathcal{T}_2, \ell_2)$ and f_o^n is the relative Ehrhard function associated to $\Delta(\mathcal{T})$ then $f_o^n = f_{\ell_1}^n \star f_{\ell_2}^n$

Proof. The definitions of \star are made accordingly. □

Example 3.36. By using 3.32 we find out that

$$\sum_{k=0}^n (1^n)^{\star 2}(k) = \frac{(n+1)(n+2)(n+3)}{6}$$

which is the Poincare-Hilbert polynomial of $(\mathbb{P}^3, \mathcal{O}(1))$ while

$$\sum_{k=0}^n (1^n)^{\star 3}(k) = \frac{(n+1)(n+2)(n+3)(n^2+4n+5)}{30}$$

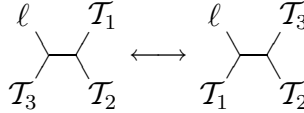
which is Poincare-Hilbert polynomial of intersection of two quadrics in \mathbb{P}^7 .

◦

Theorem 3.37. *Let us consider three pointed trees (\mathcal{T}_i, ℓ_i) , with $i = 1, 2, 3$ with relative Ehrhard functions $f_i^n = f_{\ell_i}^n$ associated to polytopes $\Delta(\mathcal{T}_i)$, respectively. Then*

$$(f_1^n \star f_2^n) \star f_3^n = f_1^n \star (f_2^n \star f_3^n)$$

Proof. Let ℓ denote the distinguished leaf of the result of the \star operation on the trees. Then the relative Ehrhard function $(f_1^n \star f_2^n) \star f_3^n$ and, respectively, $f_1^n \star (f_2^n \star f_3^n)$ is related to one of the following trees, each of them is obtained by an elementary mutation from the other:



Now we repeat the construction 3.23, with obvious modifications. Namely, we define a polytope

$$\overline{\Delta} = \widetilde{\Delta}_0 \times \prod_{i=1}^3 \Delta(\mathcal{T}_i) \cap \bigcap_{i=1}^3 \ker(\ell_i - \tilde{v}_i)$$

where $\widetilde{\Delta}_0$ is the unit simplex as in 3.5. We define a toric variety $\mathcal{Y} = X(\overline{\Delta})$ with the embedding in $\mathbb{P}(W_{\mathcal{E}}^{\rho})$ and the action of the group λ_{ℓ} .

Next, as in 3.24 we define a subvariety $\mathcal{X} \subset \mathcal{B} \times \mathcal{Y}$ such that the projection $p_{\mathcal{B}} : \mathcal{X} \rightarrow \mathcal{B}$ is flat and its two fibers are varieties associated to the above two pointed trees, see 3.25. Because of the flatness the sheaf $(p_{\mathcal{B}})_*(p_{\mathcal{Y}}^*(\mathcal{O}(n)))$ is locally free for each $n \geq 0$, see [Hartshorne, III.9.9, III.12.9] and 3.18. Moreover, by the construction, the action of the group λ_{ℓ} on \mathcal{Y} leaves $\mathcal{X} \subset \mathcal{B} \times \mathcal{Y}$, as we noted in 3.22. Finally, the decomposition into eigenspaces of the action of λ_{ℓ} on $H^0(\mathcal{Y}, \mathcal{O}(n))$ restricts into a respective eigenspace decomposition of the action of λ_{ℓ} on fibers of $(p_{\mathcal{B}})_*(p_{\mathcal{Y}}^*(\mathcal{O}(n)))$, which are equal to $H^0(\mathcal{X}_b, \mathcal{O}(n))$, for $b \in \mathcal{B}$. This implies that the dimension of the respective eigenspaces is locally constant, with respect to the parameter $b \in \mathcal{B}$ hence the relative Ehrhard function of fibers of $p_{\mathcal{B}}$ is constant which concludes the argument. \square

Let us underline the fact that although the invariance of the Hilbert polynomial is a standard property of a flat family the above result is about the invariance of the family with respect to an action of a 1-parameter group, the group λ_ℓ in our case.

The above theorem 3.37 implies that the operation \star on relative Ehrhard functions of polytopes of 3-valent trees is not only commutative (which is obvious from its definition) but also associative. This implies that the function does not depend on either the shape nor the location of the leaf. More precisely we have the following formula which allows to compute the Hilbert-Ehrhard polynomial very efficiently.

Corollary 3.38. *If (\mathcal{T}, ℓ) is a pointed 3-valent tree with $r + 1$ leaves then*

$$f_{\Delta(\mathcal{T}), \ell}^n = (1^n)^{\star r}$$

A Appendix

A.1 Normal polytopes, unimodular covers

A lattice simplex $\Delta^0 \subset M_{\mathbb{R}}$ with vertices v_0, \dots, v_r is called unimodular if vectors $v_1 - v_0, \dots, v_r - v_0$ span M . We say a lattice polytope $\Delta \subset M_{\mathbb{R}}$ has a unimodular covering if $\Delta = \bigcup_{\nu} \Delta_{\nu}^0$ where Δ_{ν}^0 are unimodular simplexes. This definition is taken from [BGT] where we also have the following result.

Lemma A.1. *If a lattice polytope $\Delta \subset M_{\mathbb{R}}$ has a unimodular covering then it is normal.*

The following observation is probably known but we include its proof because of the proof of the subsequent lemma.

Lemma A.2. *Let $\Delta_1 \subset (M_1)_{\mathbb{R}}$ and $\Delta_2 \subset (M_2)_{\mathbb{R}}$ be two unimodular simplexes. Then $\Delta_1 \times \Delta_2$ has a unimodular covering in $M_1 \times M_2$.*

Proof. We can assume that Δ_1 has vertices $0, e_1, \dots, e_r$ and Δ_2 has vertices $0, f_1, \dots, f_s$. Suppose that $x \in (M_1)_{\mathbb{R}} \times (M_2)_{\mathbb{R}}$ is as follows:

$$x = \sum_{i=1}^r a_i e_i + \sum_{j=1}^s b_j f_j$$

where $a_i, b_j \geq 0$ and $\sum a_i \leq \sum b_j \leq 1$.

The union of unimodular simplexes contained in $\Delta_1 \times \Delta_2$ is a closed subset. Therefore if x is not contained in any modular subsimplex of $\Delta_1 \times \Delta_2$ then any

small perturbation of x has this property as well. Thus we are free to assume that all a_i 's and b_j 's are nonzero and any two non-empty subsets of a_i 's and b_j 's have different sum, in particular $a_1 + \dots + a_p \neq b_1 + \dots + b_q$ for any reasonable (p, q) . Let m be such $b_1 + \dots + b_{m-1} < a_1 + \dots + a_r < b_1 + \dots + b_m$. We set $b'_m = (b_1 + \dots + b_m) - (a_1 + \dots + a_r)$.

In order to prove the lemma we will find $r + m - 1$ positive numbers $c_{i,j}$ indexed by some pairs $(i, j) \in \{1, \dots, r\} \times \{1, \dots, m\}$ such that

$$x = \sum_{ij} c_{i,j}(e_i + f_j) + b'_m f_m + b_{m+1} f_m + \dots + b_s f_s$$

and the set of respective vectors $(e_i + f_j)$ together with f_m, \dots, f_s can be modified via addition or subtraction of pairs among them to the standard basis $e_1, \dots, e_r, f_1, \dots, f_s$.

The coefficients $c_{i,j}$ are defined inductively according to the following rules. The first coefficient is $c_{1,1} = \min\{a_1, b_1\}$. Suppose that the last defined coefficient is c_{i_0, j_0} . If $(i_0, j_0) = (r, m)$ then we are done so assume that it is not the case. Then, because of our assumption that the sequences (a_i) and (b_j) have no equal partial sums, either $a_1 + \dots + a_{i_0} > b_1 + \dots + b_{j_0}$, or $a_1 + \dots + a_{i_0} < b_1 + \dots + b_{j_0}$. In the former case we set

$$c_{i_0, j_0+1} = \min\{b_{j_0+1}, (a_1 + \dots + a_{i_0}) - (b_1 + \dots + b_{j_0})\}$$

whereas in the latter case we define

$$c_{i_0+1, j_0} = \min\{a_{i_0+1}, (b_1 + \dots + b_{j_0}) - (a_1 + \dots + a_{i_0})\}$$

The verification that $\sum_{i=1}^r c_{i,j} = b_j$ for $j = 1 \dots m - 1$ and $\sum_{j=1}^m c_{i,j} = a_i$ for $i = 1, \dots, r$ is easy and left for the reader. Similarly, a simple backtracking allows to modify the set of the respective vectors $e_i + f_j$ with f_m, \dots, f_s to the standard basis for $M_1 \times M_2$. \square

Lemma A.3. *Let $\Delta_1 \subset (M_1)_{\mathbb{R}}$, $\Delta_2 \subset (M_2)_{\mathbb{R}}$ be two unimodular simplexes. We consider two homomorphisms $\ell_i : M_i \rightarrow \mathbb{Z}$ such that $(\ell_i)_{\mathbb{R}}(\Delta_i) \subset [0, 1]$. The fiber product $\Delta = (\Delta_1)_{\ell_1 \times \ell_2}(\Delta_2)$ has a unimodular covering with respect to the fiber product lattice $M = (M_1)_{\ell_1 \times \ell_2}(M_2)$.*

Proof. The argument is a variation of the one used in the previous lemma. We can assume that Δ_1 has vertices $0, e_1^0, \dots, e_{r_0}^0, e_1^1, \dots, e_{r_1}^1$ and Δ_2 has vertices

$0, f_1^0, \dots, f_{s_0}^0, f_1^1, \dots, f_{s_1}^1$ where $\ell_1(e_i^0) = \ell_2(f_j^0) = 0$ and $\ell_1(e_i^1) = \ell_2(f_j^1) = 1$ for suitable i 's and j 's. Suppose that $x \in (M_1)_{\mathbb{R}} \times (M_2)_{\mathbb{R}}$ is as follows:

$$x = \sum_{i=1}^{r_0} a_i^0 e_i^0 + \sum_{i=1}^{r_1} a_i^1 e_i^1 + \sum_{j=1}^{s_0} b_j^0 f_j^0 + \sum_{j=1}^{s_1} b_j^1 f_j^1$$

where $a_i^0, a_i^1, b_j^0, b_j^1 \geq 0$, $\sum a_i^0 + \sum a_i^1 \leq 1$, $\sum b_j^0 + \sum b_j^1 \leq 1$ and moreover $\sum a_i^1 = \sum b_j^1$. The latter conditions ensures that $\ell_1(x) = \ell_2(x)$ and it is the only condition which can not be made perturbed, as in the proof of the previous lemma.

We write $x = x_0 + x_1$ where $x_0 = \sum a_i^0 e_i^0 + \sum b_j^0 f_j^0$ and $x_1 = \sum a_i^1 e_i^1 + \sum b_j^1 f_j^1$ and we repeat the proof of A.2 for x_0 and x_1 separately. The only difference is that, because of the equality $\sum a_i^1 = \sum b_j^1$, the construction will give $r_1 + s_1 - 1$ coefficients $c_{i,j}^1$ and associated pairs of vectors $e_i^1 + f_j^1$ which will enable to write $x_1 = \sum c_{i,j}^1 (e_i^1 + f_j^1)$. Thus, clearly, the respective vectors $e_i^1 + f_j^1$ do not constitute a basis of the lattice spanned by $e_1^1, \dots, e_{r_1}^1, f_1^1, \dots, f_{s_1}^1$ but of this lattice intersected with $\ker(\ell_1 - \ell_2)$. That is, among the chosen $r_1 + s_1 - 1$ vectors $e_i^1 + f_j^1$ we have $e_1^1 + f_1^1$ and $e_{r_1}^1 + f_{s_1}^1$ and if $e_i^1 + f_j^1$ is among them then either $e_{i+1}^1 + f_j^1$ or $e_i^1 + f_{j+1}^1$ is among them as well (but not both). We are to prove that any $e_i^1 + f_j^1$ can be obtained as a sum of them. But this follows because

$$(e_i^1 + f_j^1) + (e_{i+1}^1 + f_{j+1}^1) = (e_{i+1}^1 + f_j^1) + (e_i^1 + f_{j+1}^1)$$

so any one of the above above four vectors is a combination of the other three and this observation can be used repeatedly to complete our claim. \square

Corollary A.4. *Let $\Delta_1 \subset (M_1)_{\mathbb{R}}$, $\Delta_2 \subset (M_2)_{\mathbb{R}}$ be two polytopes which have covering by unimodular simplexes. We consider two homomorphisms $\ell_i : M_i \rightarrow \mathbb{Z}$ such that $(\ell_i)_{\mathbb{R}}(\Delta_i) \subset [0, 1]$. Then the fiber product $\Delta = (\Delta_1)_{\ell_1 \times \ell_2}(\Delta_2)$ has a unimodular covering with respect to the fiber product lattice $M = (M_1)_{\ell_1 \times \ell_2}(M_2)$*

Proof. The fiber product of Δ_1 and Δ_2 is covered by fiber products of simplexes from the unimodular cover of each of them. Thus the result follows by A.3. \square

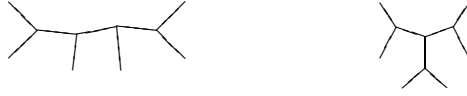
Since the polytope of the star 3-valent tree is a unit tetrahedron we get the following.

Proposition A.5. *If \mathcal{T} is a binary symmetric **3-valent** tree then its polytope in $\widehat{M}(\mathcal{T})$ has unimodular covering hence it is normal.*

A.2 Two 3-valent trees with 6 leaves

One of the fundamental questions regarding the phylogenetic trees is the following. Given two (3-valent binary symmetric) trees \mathcal{T}_1 and \mathcal{T}_2 suppose that $\Delta(\mathcal{T}_1) \cong \Delta(\mathcal{T}_2)$ as lattice polytopes, or the projective models $X(\mathcal{T}_1)$ and $X(\mathcal{T}_2)$ are projectively equivalent. Does it imply that the trees are equivalent (as CW complexes) as well?

We tackled the problem by understanding the difference of models of the two simplest non-equivalent trees. These are 6-leaf trees pictured below, respectively, a 3-caterpillar tree and a tree which we call a snow flake, [Sturmfels, Sullivan].



The snow flake tree is obtained from the 3-caterpillar tree by elementary mutation along its middle inner edge. Therefore their Hilbert-Ehrhard polynomials are equal and computed with [maxima] to be as follows.

$$h(n) = \frac{1}{22680} (n+1)(n+2)(n+3) \cdot (31n^6 + 372n^5 + 1942n^4 + 5616n^3 + 9511n^2 + 8988n + 3780)$$

On the other hand we can distinguish their polytopes in terms of some combinatorial invariants.

Example A.6. Given a polytope Δ we define its incidence matrix (a_{ij}) as follows: (a_{ij}) is a symmetric matrix with integral entries such that for $i \leq j$ the number a_{ij} is equal to the number of i -dimensional faces contained in j -dimensional faces of Δ . In particular a_{ii} is the number of i -dimensional faces. The following is the incidence matrix of a polytope of the snow flake tree.

| | | | | | | | | |
|------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|-------------|
| 32 | 480 | 2400 | 6144 | 9312 | 8832 | 5280 | 1920 | 384 |
| 480 | 240 | 2400 | 9456 | 19920 | 24960 | 19200 | 8880 | 2256 |
| 2400 | 2400 | 760 | 5944 | 19008 | 32552 | 32408 | 18792 | 5872 |
| 6144 | 9456 | 5944 | 1316 | 8400 | 21744 | 29308 | 21720 | 8388 |
| 9312 | 19920 | 19008 | 8400 | 1392 | 7200 | 14640 | 14640 | 7200 |
| 8832 | 24960 | 32552 | 21744 | 7200 | 940 | 3820 | 5760 | 3820 |
| 5280 | 19200 | 32408 | 29308 | 14640 | 3820 | 406 | 1224 | 1224 |
| 1920 | 8880 | 18792 | 21720 | 14640 | 5760 | 1224 | 108 | 216 |
| 384 | 2256 | 5872 | 8388 | 7200 | 3820 | 1224 | 216 | 16 |

And this is the incidence matrix of the polytope of a 3-caterpillar tree.

| | | | | | | | | |
|------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|-------------|
| 32 | 480 | 2400 | 6144 | 9312 | 8832 | 5280 | 1920 | 384 |
| 480 | 240 | 2400 | 9456 | 19904 | 24896 | 19104 | 8816 | 2240 |
| 2400 | 2400 | 760 | 5944 | 18976 | 32408 | 32168 | 18616 | 5824 |
| 6144 | 9456 | 5944 | 1316 | 8384 | 21648 | 29112 | 21552 | 8336 |
| 9312 | 19904 | 18976 | 8384 | 1392 | 7184 | 14584 | 14576 | 7176 |
| 8832 | 24896 | 32408 | 21648 | 7184 | 940 | 3816 | 5752 | 3816 |
| 5280 | 19104 | 32168 | 29112 | 14584 | 3816 | 406 | 1224 | 1224 |
| 1920 | 8816 | 18616 | 21552 | 14576 | 5752 | 1224 | 108 | 216 |
| 384 | 2240 | 5824 | 8336 | 7176 | 3816 | 1224 | 216 | 16 |

Both matrices were computed by [polymake]. We note that although both polytopes have the same number of faces of respective dimension their incidences are different (indicated in boldface).

◦

A.3 Volume distribution

The leading coefficient in the Ehrhard polynomial of a lattice polytope Δ can be identified as the volume of Δ (with respect to the lattice in question, whose unit cube is assumed to have volume 1). Similarly, we can define a relative volume function which will measure the distribution of the volume of $\Delta(\mathcal{T})$ with respect to a leaf ℓ of \mathcal{T} . Because of 3.38 this function does not depend either on the shape of the tree nor on the choice of the leaf ℓ . Moreover we will normalize it so that its integral over the unit segment is 1.

If $\delta^r : [0, 1] \rightarrow \mathbb{R}$ is the normalized volume distribution with respect to a leaf of a 3-valent tree with r leaves then because of 3.31 we have $\delta^n(t) = \delta^n(1 - t)$ and for $t \in (0, 1/2)$ we get the following recursive formula

$$\delta^{r+1}(t) = d_{n+1} \cdot \left(2 \cdot \int_0^t s \cdot \delta^r(s) ds + \int_t^{1-t} t \cdot \delta^r(s) ds \right)$$

where d_{n+1} is a constant such that $\int_0^1 \delta^{r+1}(s) ds = 1$. From this it follows that δ^r is a polynomial of degree $2r$. However, the numerical experiments which we have made seem to indicate that for $r > 3$ the actual values of δ^r do not depend too much on r , see Fig. 1. It seems that this function does not see the shape of the tree (which is because it comes from the relative Hilbert-Ehrhard polynomial) but also almost disregards its size (or dimension of the model)

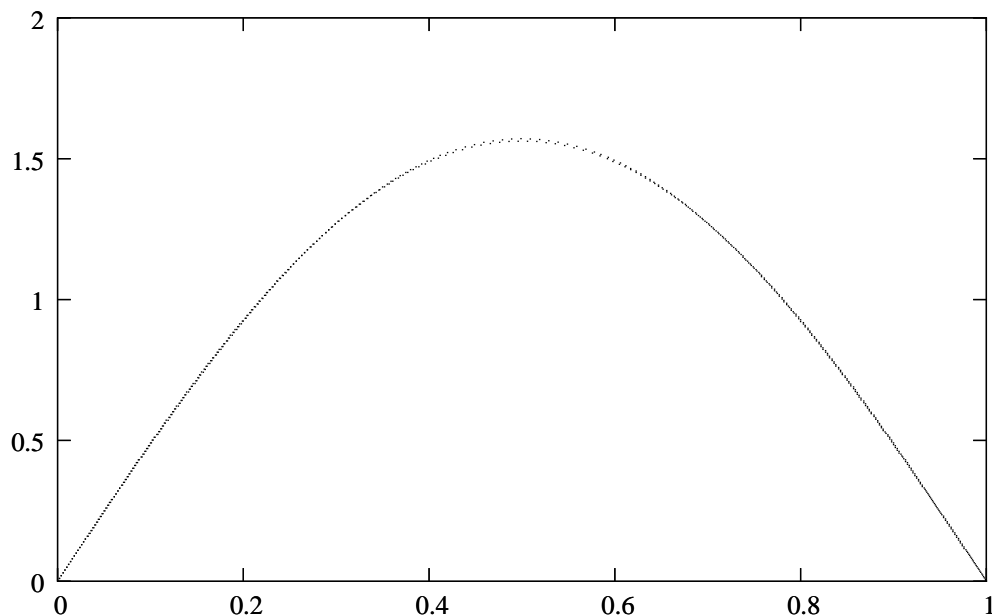


Figure 1: Polynomials δ^2 and δ^{100} at the same diagram, by [gnuplot].

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